

Finally, we point out that the enumeration of the electromagnetic modes presented here is mathematically identical to the calculation of the allowed quantum states of electrons confined within perfectly reflecting walls. The latter model is of importance in determining the density of allowed electron states as a function of energy in a semiconductor material (see Sec. 15.1C).

9.2 SPHERICAL-MIRROR RESONATORS

The planar-mirror resonator configuration discussed in the preceding section is highly sensitive to misalignment. If the mirrors are not perfectly parallel, or the rays are not perfectly normal to the mirror surfaces, they undergo a sequence of lateral displacements that eventually causes them to wander out of the resonator. Spherical-mirror resonators, in contrast, provide a more stable configuration for the confinement of light that renders them less sensitive to misalignment under certain geometrical conditions.

A spherical-mirror resonator is constructed of two spherical mirrors of radii R_1 and R_2 separated by a distance d (Fig. 9.2-1). The centers of the mirrors define the optical axis (z axis), about which the system exhibits circular symmetry. Each of the mirrors can be concave ($R < 0$) or convex ($R > 0$). The planar-mirror resonator is a special case for which $R_1 = R_2 = \infty$. We first examine the conditions for the confinement of optical rays. Then we determine the resonator modes. Finally, the effect of finite mirror size is discussed briefly.

A. Ray Confinement

Our initial approach is to use ray optics to determine the conditions of confinement for light rays in a spherical-mirror resonator. We consider only meridional rays (rays lying in a plane that passes through the optical axis) and limit ourselves to paraxial rays (rays that make small angles with the optical axis). The matrix-optics methods introduced in Sec. 1.4, which are valid only for paraxial rays, are used to study the trajectories of rays as they travel inside the resonator.

A resonator is a periodic optical system, since a ray travels through the same system after a round trip of two reflections. We may therefore make use of the analysis of periodic optical systems presented in Sec. 1.4D. Let y_m and θ_m be the position and inclination of an optical ray after m round trips, as illustrated in Fig. 9.2-2. Given y_m and θ_m , y_{m+1} and θ_{m+1} can be determined by tracing the ray through the system.

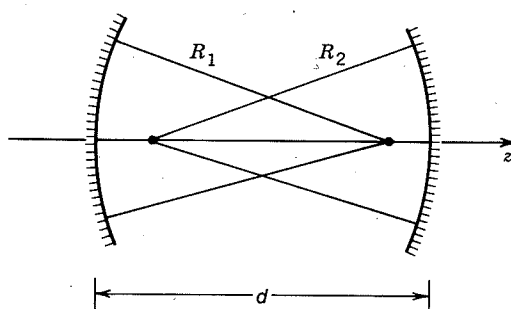


Figure 9.2-1 Geometry of a spherical-mirror resonator. In this case both mirrors are concave (their radii of curvature are negative).

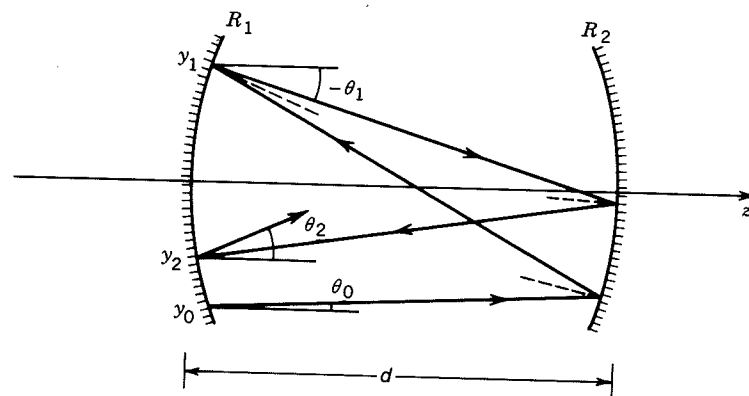


Figure 9.2-2 The position and inclination of a ray after m round trips are represented by y_m and θ_m , respectively, where $m = 0, 1, 2, \dots$. In this diagram, $\theta_1 < 0$ since the ray is going downward.

For paraxial rays, where all angles are small, the relation between (y_{m+1}, θ_{m+1}) and (y_m, θ_m) is linear and can be written in the matrix form

$$\begin{bmatrix} y_{m+1} \\ \theta_{m+1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_m \\ \theta_m \end{bmatrix}. \quad (9.2-1)$$

The round-trip ray-transfer matrix for Fig. 9.2-2:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2}{R_1} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{2}{R_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

is a product of ray-transfer matrices representing, from right to left [see (1.4-3) and (1.4-8)]:

- propagation a distance d through free space,
- reflection from a mirror of radius R_2 ,
- propagation a distance d through free space,
- reflection from a mirror of radius R_1 .

As shown in Sec. 1.4D, the solution of the difference equation (9.2-1) is $y_m = y_{\max} F^m \sin(m\varphi + \varphi_0)$, where $F^2 = AD - BC$, $\varphi = \cos^{-1}(b/F)$, $b = (A + D)/2$, and y_{\max} and φ_0 are constants to be determined from the initial position and inclination of the ray. For the case at hand $F = 1$, so that

$$y_m = y_{\max} \sin(m\varphi + \varphi_0), \quad (9.2-2)$$

$$\varphi = \cos^{-1} b, \quad b = 2 \left(1 + \frac{d}{R_1} \right) \left(1 + \frac{d}{R_2} \right) - 1.$$

The solution (9.2-2) is harmonic (and therefore bounded) provided that $\varphi = \cos^{-1} b$ is real. This is ensured if $|b| \leq 1$, i.e., if $-1 \leq b \leq 1$ or $0 \leq (1 + d/R_1)(1 + d/R_2) \leq 1$. It is convenient to write this condition in terms of the parameters $g_1 = 1 + d/R_1$ and

$g_2 = 1 + d/R_2$, which are known as the g parameters,

$$0 \leq g_1 g_2 \leq 1.$$

(9.2-3)

Confinement Condition

When this condition is not satisfied, φ is imaginary so that y_m in (9.2-2) becomes a hyperbolic sine function of m which increases without bound. The resonator is then said to be **unstable**. At the boundary of the confinement condition (when the inequalities are equalities), the resonator is said to be **conditionally stable**; slight errors in alignment render it unstable.

A useful graphical representation of the confinement condition (Fig. 9.2-3) identifies each combination (g_1, g_2) of the two g parameters of a resonator as a point in a g_2 versus g_1 diagram. The left inequality in (9.2-3) is equivalent to $\{g_1 \geq 0 \text{ and } g_2 \geq 0\}$; or $g_1 \leq 0$ and $g_2 \leq 0$; i.e., all stable points (g_1, g_2) must lie in the first or third quadrant. The right inequality in (9.2-3) signifies that stable points (g_1, g_2) must lie in a region bounded by the hyperbola $g_1 g_2 = 1$. The unshaded area in Fig. 9.2-3 represents the region for which both inequalities are satisfied, indicating that the resonator is stable.

Symmetrical resonators, by definition, have identical mirrors ($R_1 = R_2 = R$) so that $g_1 = g_2 = g$. The condition of stability is then $g^2 \leq 1$, or $-1 \leq g \leq 1$, so that

$$0 \leq \frac{d}{(-R)} \leq 2.$$

(9.2-4)

Confinement Condition
(Symmetrical Resonator)

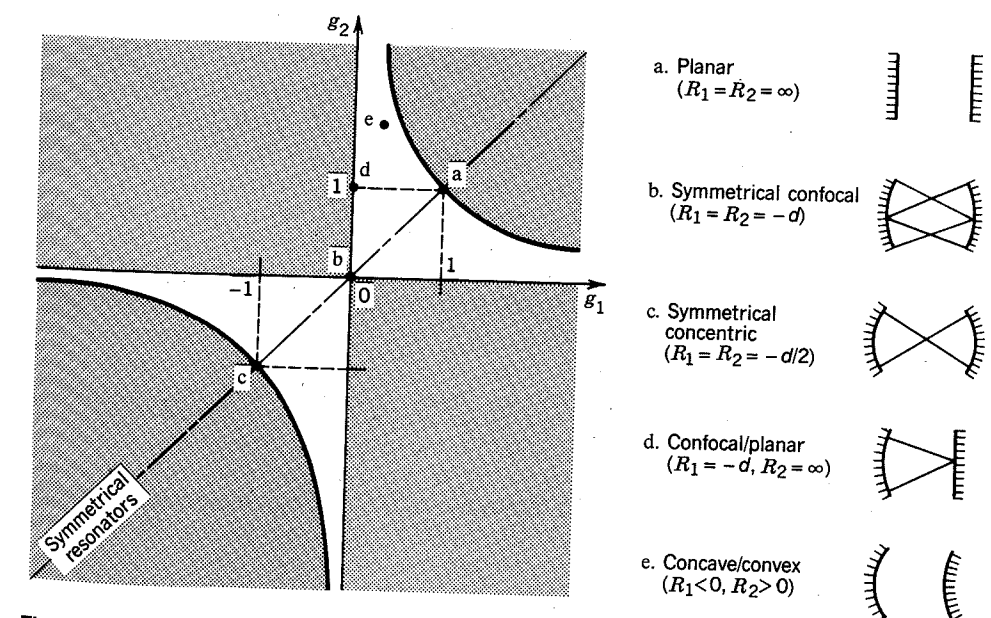


Figure 9.2-3 Resonator stability diagram. A spherical-mirror resonator is stable if the parameters $g_1 = 1 + d/R_1$ and $g_2 = 1 + d/R_2$ lie in the unshaded regions bounded by the lines $g_1 = 0$ and $g_2 = 0$, and the hyperbola $g_2 = 1/g_1$. R is negative for a concave mirror and positive for a convex mirror. Various special configurations are indicated by letters. All symmetrical resonators lie along the line $g_2 = g_1$.

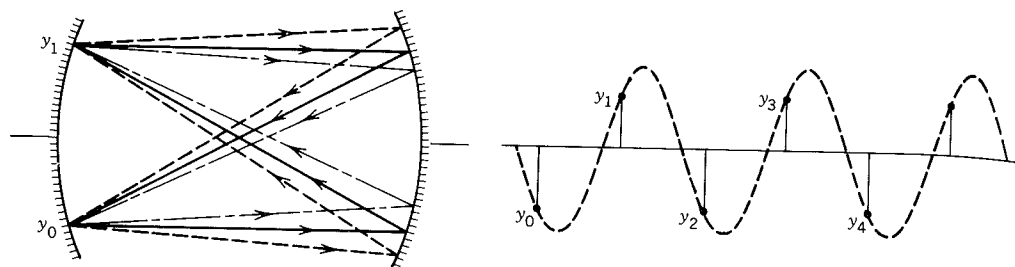


Figure 9.2-4 All paraxial rays in a symmetrical confocal resonator retrace themselves after two round trips, regardless of their original position and inclination. Angles are exaggerated in this drawing for the purpose of illustration.

These resonators are represented in Fig. 9.2-3 by points along the line $g_2 = g_1$. To satisfy (9.2-4) a stable symmetrical resonator must use concave mirrors ($R < 0$) whose radii are greater than half the resonator length. Three points within this interval are of special interest: $d/(-R) = 0, 1$, and 2 , corresponding to **planar**, **confocal**, and **concentric** resonators, respectively.

In the symmetrical confocal resonator, $(-R) = d$, so that the center of curvature of each mirror lies on the other. Thus in (9.2-2), $b = -1$, $\varphi = \pi$, and the ray position is $y_m = y_{\max} \sin(m\pi + \varphi_0)$, i.e., $y_m = (-1)^m y_0$. Rays initiated at position y_0 , at any inclination, are imaged to position $y_1 = -y_0$, then imaged again to position $y_2 = y_0$, and so on, repeatedly. Each ray retraces itself after two round trips (Fig. 9.2-4). All paraxial rays are therefore confined, no matter what their original position and inclination. This is to be compared with the planar-mirror resonator, for which only rays of zero inclination retrace themselves.

In summary, the confinement condition for paraxial rays in a spherical-mirror resonator, constructed of mirrors of radii R_1 and R_2 separated by a distance d , is $0 \leq g_1 g_2 \leq 1$, where $g_1 = 1 + d/R_1$ and $g_2 = 1 + d/R_2$.

EXERCISE 9.2-1

Maximum Resonator Length for Confined Rays. A resonator is constructed using concave mirrors of radii 50 cm and 100 cm. Determine the maximum resonator length for which rays satisfy the confinement condition.

B. Gaussian Modes

Although the ray-optics approach considered in the preceding section is useful for determining the geometrical conditions under which rays are confined, it cannot provide information about the spatial intensity distributions and resonance frequencies of the resonator modes. We now proceed to show that Gaussian beams are modes of