"Solution for Electromagnetic waves tutorial"

0) As explained in example, to find the possible solution, we should substitute the function in the differential equation and try to find some meaningful relation between the Constants of the problem.

For $x = At^2$, we end up with $-KAt^2 = 2Am$, that makes no sense because the right hand side is a Constant where the left hand side is time dependent.

In Contrast, for $x = A\sin(Bt)$ we end up with $-kA\sin(Bt) = -mAB^2\sin(Bt)$. After simplifying the relation, we have $B^2 = \frac{k}{m}$. It means that if we have this relation between Constants of the function, it Could be a reasonable solution for any time.

1) a) $E(x,t) = A\cos(Bt)$

By substituting it in differential equation, we have:

$$\frac{\partial^2 E}{\partial t^2} = -AB^2\cos(Bt) \quad , \quad \frac{\partial^2 E}{\partial x^2} = 0$$

$$\Rightarrow \quad AB^2\cos(Bt) = 0$$

if we choose A or B equal to zero, we will find a trivial solution. Additionally, we have a Constant side and a time-dependent side. So, this function Could not be a solution.

b) $E(x,t) = Ax^2t^2 \Rightarrow$

$$\frac{\partial^2 E}{\partial t^2} = 2Ax^2 \quad , \quad \frac{\partial^2 E}{\partial x^2} = 2At^2$$

So, we have: $$2At^2 = \frac{1}{c^2} \times 2Ax^2$$

This relation does not make any senge because, x and t are independent variables and can not have such a relationship. Beside that, if you use this relation to rewrite the electric field, we have:

$$E(x,t) = Ac^2t^4 \quad \text{or} \quad E(x,t) = \frac{A}{c^2}x^4$$

that again does not make sense, since they grow up in time and space monotically and diverge.

c) $E(x,t) = A\cos(Bx + Ct)$ $\Rightarrow$

$$\frac{\partial^2 E}{\partial x^2} = -AB^2\cos(Bx+Ct) \quad , \quad \frac{\partial^2 E}{\partial t^2} = -AC^2\cos(Bx+Ct)$$

together we have:

$$-AB^2\cos(Bx+Ct) = \frac{1}{c^2} \times -AC^2\cos(Bx+Ct)$$

$$\Rightarrow \quad B^2 = \frac{C^2}{c^2}$$

It means that if we consider this relation between our constants, $E(x,t) = A\cos(Bx+Ct)$ could be a possible solution. And the constraint between constants is: $C^2 = c^2B^2$

2) We know that $E(x,t) = A\cos(Bx+Ct)$ has 2π period. It means that if you add $2\pi n$ to the argument of the cosine function, it would not change.

To find the period in time, we should look at ct term.

We have: $cT = 2\pi$ where T is the period

$\Rightarrow \quad T = \frac{2\pi}{c} \quad \left(\cos(Bx + ct) = \cos(Bx + ct + cT) \right)$

Frequency simply could be found from $f = \frac{1}{T}$

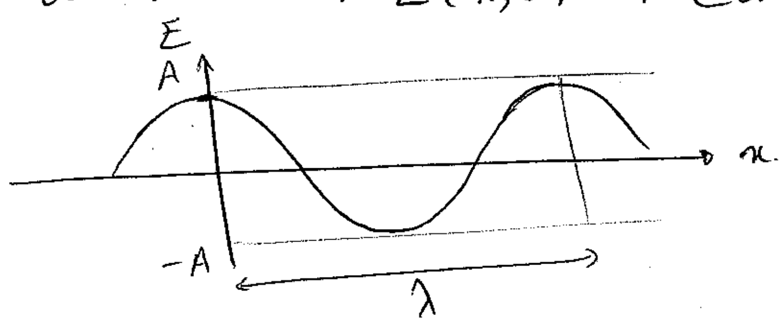
$\Rightarrow \quad f = \frac{c}{2\pi}$

3) This time we should look at Bx term. So, we have:

$$\cos(Bx + ct) = \cos(Bx + B\lambda + ct)$$

or $\quad B\lambda = 2\pi \quad \Rightarrow \quad \lambda = \frac{2\pi}{B}$ where λ is the wave length.

4) for $t = 0$ we have: $E(x, 0) = A\cos(Bx)$

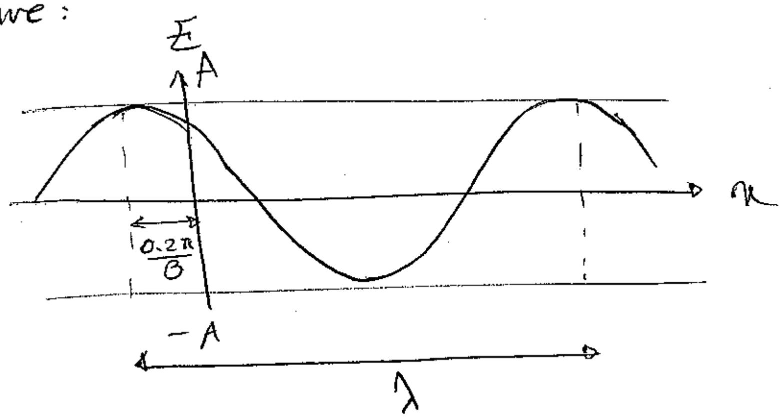


for $t = 0.2\,\pi/c$ we have: $E(x, 0.2\pi/c) = A\cos(Bx + 0.2\pi)$

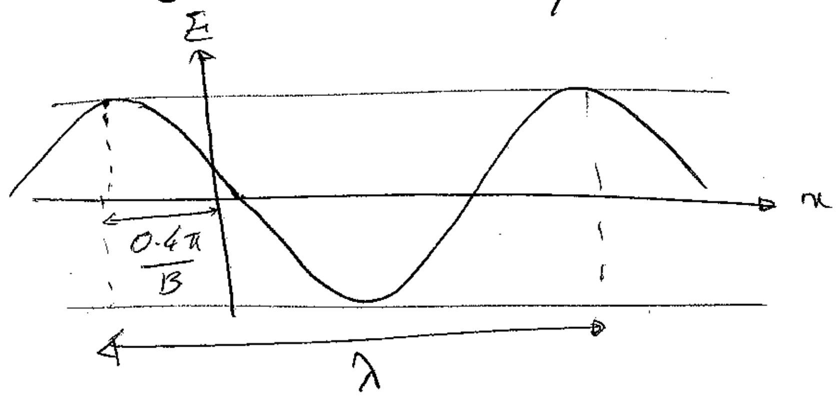
Here we have the same graph as above but shifted by $\frac{0.2\pi}{B}$ to left because, we can write E as:

$$E(x, 0.2\pi/c) = A\cos\left(B\left(x + \frac{0.2\pi}{B}\right)\right)$$

So, we have:



and for $t = 0.4\pi/C$ we have similarly:



By looking at above plots, it shows that the wave travels toward left side of the page. By speed of the wave we mean the speed at which wave travel distance λ (one full wavelength) in time interval T (one full period). By using part 2) and 3) we simply can write:

$$V \equiv \frac{\lambda}{T} = \frac{\frac{2\pi}{B}}{\frac{2\pi}{C}} = \frac{C}{B} = c$$

5) The other possible solution would be,

$$E(x,t) = A' \cos(B'x - C't) \qquad \text{(notice the minus sign!)}$$

If, we substitute this function in differential equation, we will find the same constraint as part c) of question 1) : $B'^2 = \frac{C'^2}{c^2}$

Notice that because of the minus sign this wave travels to right side of the page. So, the general solution would be the linear Combination of these two solutions:

$$E(x,t) = A\cos(Bx + Ct) + A'\cos(B'x - C't)$$

please note that because our differential equation is linear, any linear Combination with arbitrary Coefficients would be the solution. So, in general the two amplitudes are not the Same.

6) Now, we want to Consider boundary Condition that tell us the electric field at both ends of the microwave oven is zero. This Condition is exactly the same as a wave on a string when you fix both ends of the string. Wave reaches the boundaries and it reflects from the walls and interferes with another wave that moves in opposite direction. So, we would end up with standing waves. Note that because left-going and right-going waves have the same nature (one is the reflection of the other), we Consider the same wavelength and frequency

for both of them. It means in general solution in problem 5) we consider $B = B'$ and $C = C'$.

Now, let's check the boundary conditions:

I) $E(x=0) = 0 \Rightarrow A\cos(ct) + A'\cos(-ct) = 0$

Cosine, is an odd function, So, $\cos(x) = \cos(-x)$

$$\Rightarrow (A + A')\cos(ct) = 0$$

Because we want to find the solution for all the time, the only choice is: $A + A' = 0$ or $A = -A'$

$$\Rightarrow E(x,t) = A\left(\cos(Bx + Ct) - \cos(Bx - ct)\right)$$

II) $E(L = 0.59m) = 0$

Let me first simplify the above equation by using trigo identities

$$\cos(s+t) = \cos(s)\cos(t) - \sin(s)\sin(t)$$
$$\cos(s-t) = \cos(s)\cos(t) + \sin(s)\sin(t)$$

So, we would have:

$$E(x,t) = A\left[\cos(Bx)\cos(ct) - \sin(Bx)\sin(ct) - \cos(Bx)\cos(ct) - \sin(Bx)\sin(ct)\right]$$

$$= -2A\sin(Bx)\sin(ct)$$

Now, for $x = L$ we have:

$$2A \sin(BL) \sin(Ct) = 0$$

$A = 0$ would be trivial solution. So, we ignore it.

The other possible equation to have a solution for all the times would be: $\sin(BL) = 0$

$$\Rightarrow BL = n\pi \Rightarrow B = \frac{n\pi}{L} = \frac{n\pi}{0.59}$$

for $n = \pm 1, \pm 2, \dots$

from $B^2 = \frac{C^2}{c^2}$ we can find C:

$$C^2 = c^2 B^2 = \frac{c^2 n^2 \pi^2}{(0.59)^2} \Rightarrow C = \frac{c n \pi}{0.59} \quad \text{for } n = \pm 1, \pm 2, \dots$$

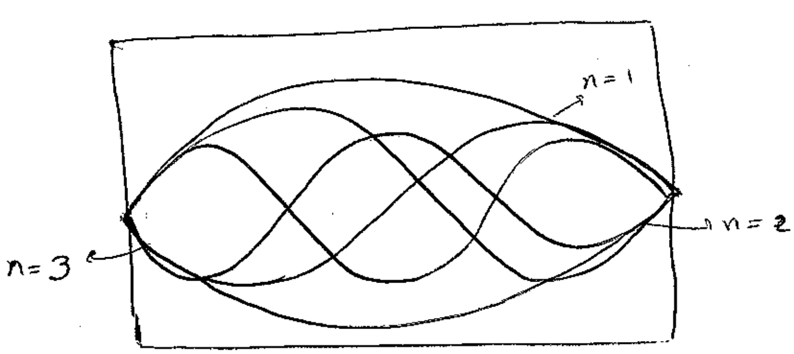
We can simply find the constraint on wavelength by using $\lambda = \frac{2\pi}{B}$

$$\Rightarrow \lambda = \frac{2\pi}{n\pi/0.59} = \frac{2 \times 0.59}{n} \quad \text{or} \quad \frac{n\lambda}{2} = 0.59 \text{ m}$$

$n = 1, 2, \dots$

That is the famous relation that you know for standing waves.

Here you can see the wave for $n = 1, 2, 3$ for example.



7) To find the wavelength we use the definition of speed of the wave in problem 4) : $c = \frac{\lambda}{T}$ or $c = \lambda f$

Where here c is the speed of electomagnetic waves that is equal to:

$$c = 3 \times 10^8 \ m/s$$

$$\Rightarrow \quad \lambda = \frac{c}{f} = \frac{3 \times 10^8}{2.54 \times 10^9} = 0.118 \ m$$

The length of the microwave oven is $0.59 \ m$ and $\frac{0.59}{0.118} = 5$

It means that we have 5 full wavelengthes in our standing wave.

The electric field at different times has following equation:

$$E(x,t) = -2A \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{cn\pi}{L}t\right)$$

from $\frac{n\lambda}{2} = L$ we have $n = 10$.

$$\Rightarrow \quad E(x,t) = -2A \sin\left(\frac{10\pi}{L}x\right) \sin\left(\frac{10c\pi}{L}t\right)$$

There is a mistake in typing for different time slices. because of the high frequency we have a very big c. So, to see the change in the wave we consider the waves at times:

$$t = 0, \quad t = \frac{0.25\pi}{c}, \quad t = \frac{0.5\pi}{c}, \quad t = \frac{0.75\pi}{c}, \quad t = \frac{\pi}{c}.$$

At $t=0$ we have: $E(x,t) = 0$.

At $t = \frac{0.25\pi}{c}$, $E(x,t) = -2A\sin\left(\frac{10\pi}{L}x\right)\sin(0.25\pi)$

It means that we have a wave in x-direction with the amplitude equal to $2A\sin(\pi/4) = \sqrt{2}A$

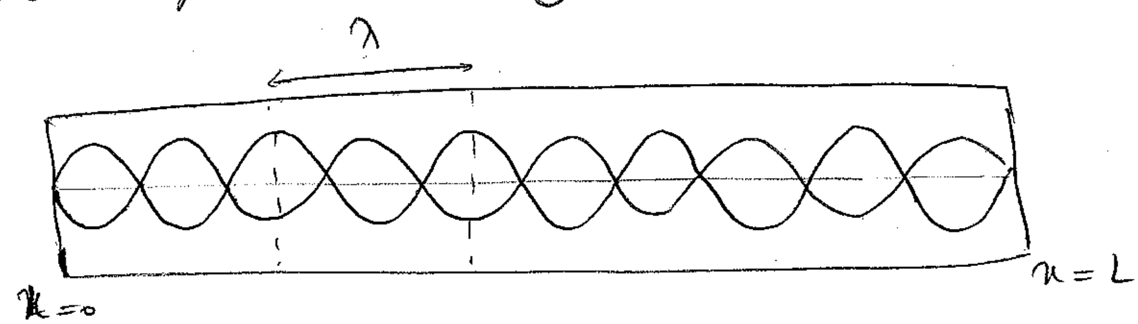
At $t = \frac{0.5\pi}{c}$, $E(x,t) = -2A\sin\left(\frac{10\pi}{L}x\right)\sin\left(\frac{\pi}{2}\right)$

with the amplitude $2A\sin\left(\frac{\pi}{2}\right) = 2A$

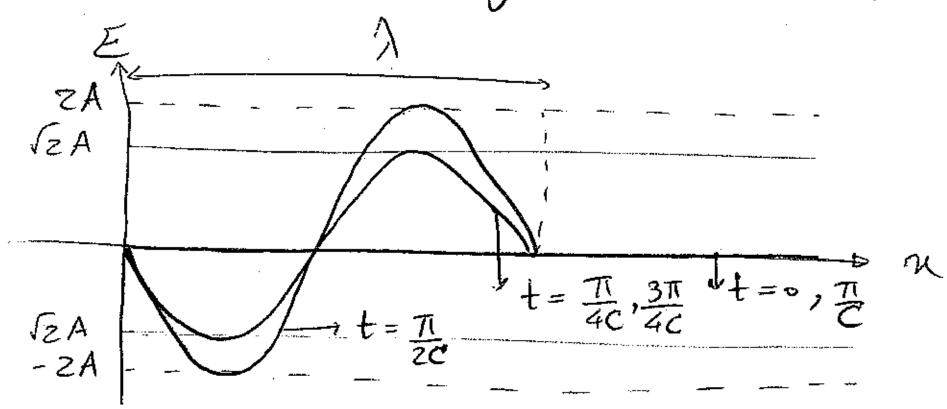
$t = \frac{0.75\pi}{c} \Rightarrow E(x,t) = -2A\sin\left(\frac{10\pi}{L}x\right)\sin\left(\frac{3\pi}{4}\right)$

$t = \frac{\pi}{c} \Rightarrow E(x,t) = -2A\sin\left(\frac{10\pi}{L}x\right)\sin(\pi) = 0$

The typical shape of the standing waves would be:



But for wave at above different times, for simplicity I only draw the first full wave length. But the true picture has 5 wavelengths.



To find the period, it is good to look at one of the maximums in the wave. We define period as the time for a point at maximum, to reach the same electric field after evolution of the wave. In above picture, you can see only half of the period evolution. Note that our picture is consistent with boundary conditions. At both sides, electric field is zero.

In above picture you can see the point at $x=\frac{\lambda}{2}$ that always has zero electric field. We have the same condition for all $x=\frac{n\lambda}{2}$. Since electric field is zero at these points, there is no effect on the molecules at these points. So, at these particular locations in microwave oven, the food does not heat as well as at other locations.

8) For this problem, we have the same boundary condition with different size of the system. So, again, we have standing waves. To find the number of half wavelengths fit between the mirrors we use: $\frac{n\lambda}{2}=L \Rightarrow n=\frac{2L}{\lambda}=\frac{2\times 2}{541.5\times 10^{-9}}$

$\Rightarrow n = 7386890$!