Electromagnetic waves tutorial

This is the first part of a two part long answer question.
It will be to your benefit to read the example through more than once to understand what you are doing fully.

Basically this problem is for you to start from what Maxwell's equations say about time dependence of electric fields and go through to solve what the oscillating electric field looks like inside a microwave oven and a laser cavity. If you do that, you can in principle pretty much solve any problem involving electromagnetic fields. However, we were afraid that this might be too hard a problem to just ask you with no help, so we have given you an example that illustrates all the math and physics steps you need to follow and we have broken down the solution into the different steps that are required. This problem also serves as a "minicourse" covering the differential equations you will need later in the course. You should read all the way through this problem, including the worked example at the beginning, before you start trying to solve it. Doing so will help you see how the pieces fit together and make solving it easier. This problem deals with the basic properties of light. Skipping a hundred years or so of experiment and argument, it is now well established that light, and all electromagnetic radiation, are coupled oscillating electric and magnetic fields. In this course we will concentrate primarily on the electric part because that is what exerts a force on the electron, which is the primary mechanism by which light interacts with atoms.

It was well established in the 19th century that light behaves as a classical wave, with the most compelling experiment being that the interference pattern observed when light passed through a double slit was identical to that observed with sound waves or water waves as they pass through a double slit. However, to get a microscopic view of the electric field in a light wave we must use the fact that the fields obey Maxwell's equations and then calculate from these equations what the electric field does as a function of space and time. Once we know that, we can then use that information to start understanding how that electric field interacts with matter by exerting forces on the electrons and protons.

With a bit of algebra, one can show that Maxwell's equations describing electric and magnetic fields gives that the vector field **E** is given by:

$$\nabla^2 E = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

which is a highly condensed form of a long equation. Taking just the 1-dimensional case, which is a lot simpler and all that we will need to worry about in this class, this equation reduces to:

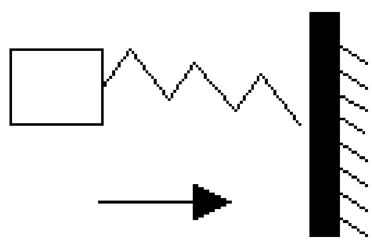
$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} \quad \text{(EM wave equation.)}$$

This just says that the second derivative of the electric field with respect to x is equal to a constant times the second derivative of **E** with respect to time. If you are not familiar with the symbol, , it means a "partial derivative". Just think of it as like the derivative symbol d, but it is used when there is a quantity like **E** that is a function of more than one variable. Here **E** is a function of both position and time.

So the problem is now to solve this equation to see what **E(x,t)** is. Later in the course we will need to solve other equations that are similar to this one, so you should remember the steps involved in the solution so you can use them again. If you have already studied differential equations, you can do this problem quickly, since it is a differential equation (any equation with a derivative in it).
However, we are going to assume you have not had differential equations, and so we will first show you how to solve a similar problem involving something you are familiar with, a mass attached to a spring. Then you will need to apply the same process to solve the equation for the electric field of a light wave.

Example problem illustrating steps required in solving differential equation.

Worked example: To find the position as a function of time [x(t)] of a mass of given size m that is attached to a spring of known spring constant k, after it is pulled back 1 cm and then released.



As you should remember from earlier physics courses, the force on the mass due to the spring is **F = -kx,** but Newton's laws also tell us that **F = ma**, so we have **–kx=ma** and plugging in for $a = \frac{\partial x^2}{\partial t^2}$ we have that:

This is the differential equation governing the position of the mass, *x(t)*. There are basically two ways to solve differential equations like this:

1) Go ask a computer the answer, and 2) Guess the solution. In this course, as in differential equations classes, we will use the second approach most of the time. So to solve this, we first try to guess a suitable algebraic function for *x* that satisfies this equation.

A) Guessing and testing solutions to the differential equation

Let's try $\mathbf{x = At^2}$ (where A is an arbitrary constant to be determined) as a trial solution. Plugging in to the above equation and taking the derivative we get $\mathbf{-kAt^2 = m(2A)}$.
This equation makes no sense, because on the left side of the equation we have the variable **t,** and on the right we only have constants. So that means that $\mathbf{x = At^2}$ does not work as a solution.

Let's try another guess, $\mathbf{x = A\sin(Bt)}$. Plugging this in we get $\mathbf{-kA\sin(Bt)}$ on the left and $\mathbf{-mAB^2\sin(Bt)}$ on the right. So $\mathbf{-kA\sin(Bt) = -mAB^2\sin(Bt)}$ **.** Does this equation make sense? The **A'**s **and sin(Bt)'**s divide out leaving $\mathbf{k = mB^2}$**.** Since these are all constants, if $\mathbf{B^2 = k/m}$**,** this equation works. That means that $\mathbf{x = A\sin(Bt)}$ is a solution to this differential equation, but only if we constrain **B**. We see that $\mathbf{B^2}$ must be **= k/m.**
This is a solution, but it is not the only solution, and hence may not describe the position of the mass as a function of time for all situations. We might guess another possible solution would be $\mathbf{x = C\cos(Dt)}$**.** You should plug in to check that this also works as a solution as long as $\mathbf{D^2 = k/m}$**.**

B) Interpreting the behavior of solutions (finding the period and frequency).

This solution says that the value of x increases and decreases periodically as a function of time. The time it takes to repeat is the period, and in this time Dt must change by 2π radians. So the period is T = 2π/D seconds/cycle. The frequency is how many times the position x(t) oscillates through a full cycle per second, and so this is just 1 second/(the period). So to check your understanding, show that the frequency of oscillation f = D/2π .

C) Developing a general solution

So we see that: $x = A\sin\left[\sqrt{k/m} * t\right]$

is a solution, but so is $x = C * \cos\left[\sqrt{k/m} * t\right]$ as well.

How are you supposed to know which solution to use? The most general possible solution is the sum (or superposition) of these two: $x = A\sin\left[\sqrt{k/m} * t\right] + C * \cos\left[\sqrt{k/m} * t\right]$

You can check for yourself: if you have two functions that work as a solution to a linear differential equation when you plug them in, then plugging in their sum will also work as a solution.

So it may seem like this could go on forever with you just guessing an infinite number of solutions that work, but theory of differential equations tells us things are not so bad. If we only have two derivatives in the differential equation, there are only two independent solutions, so once we have guessed both sine and cosine and see that they both work, their sum is the most general solution we can have. So we have figured out what the functional form of x(t) must be.

D) Going from a general to a specific solution: Applying boundary conditions
The next step is to figure out what those arbitrary constants A and C are. If you look at the form of the solution, you realize that A and C will describe where the mass will be and how fast it will be moving at any particular time. So these constants are determined by what we call "the boundary conditions or values" of the solution, namely what value the solution is at some particular time and/or place. This value is set by the physical situation that is specified. So if we are given no other information about the position of this mass, there is no way we can figure out what they must be.

However, if we are given more information, such as the mass is pulled to stretch the spring a distance of 1 cm and it is released with zero velocity at **t=0**. The constants A and C must be the correct values to describe this situation. So at **t=0**,
the solution must be **x(t=0) = -1cm** and **v(t=0) =0**. So we use our general solution and these specific requirements: $x = Asin\left[\sqrt{k/m}^{*} t\right] + C^{*} \cos\left[\sqrt{k/m}^{*} t\right]$

To apply the first boundary condition, **x(t=0) = -1cm**, we just plug in **t=0** and **x = -1cm:**
-1 cm = Asin(0) +Dcos(0).

If you think about this a little, you should be able to convince yourself that the only way to satisfy this condition is for **D=-1 cm.** The value of **A** could be any value and still have the position correct at that time. To find A, we need to apply the second boundary condition, **v(t=0) = 0.** To do this, we need to first take the time derivative of the general solution to get an equation for v:

$$v = \sqrt{k/m}^{*} Acos\left[\sqrt{k/m}^{*} t\right] - \sqrt{k/m}^{*} Csin\left[\sqrt{k/m^{*} t}\right]$$

Then we plug in **t=0** and **v=0:** $0 = \sqrt{k/m} * A\cos[0] - \sqrt{k/m} * C\sin[0]$

You should be able to convince yourself that the only way to satisfy this condition is for **A=0.**

So finally we have a solution to our differential equation that describes the position of our mass at any subsequent time: $x = -1cm * \cos\left[\sqrt{k/m} * t\right]$

For slightly more complex differential equations, such as you will be encountering, there will be more than two constants, and you will need to know additional information (more "boundary conditions") about the system in order to figure out what the constants must be for any particular physical situation.

To review this example, we figured out the behavior of the mass by:

1) Finding the differential equation that governs the mass by applying the physics that we know describes it.
2) Finding a general solution to the differential equation by guessing trial solutions that have particular functional forms of the variables in the problem and plugging them into the differential equation to see if they make sense. They make sense if there are some possible values for the unknown constants that might work, and you do not have variables equal to constants. To know you have the correct solution, you must guess a number of independent solutions and the most general solution is the sum of these. (You need 2 if you have 2 derivatives in your equation).

3) Finding the specific solution to the differential equation by using the boundary conditions that are given for the situation to determine the values that the unknown constant must have.

Long Answer Homework problem: part 1 of 2 (part 2 will be theLong Answer Homework problem: part 1 of 2 (part 2 will be the long answer assignment for homework 2) (IMPORTANT: You will want to keep in mind the rubric that will be used for grading this problem.)

1. **Identifying the physical principles or "key ideas" (expressed in words) that apply to the problem and your strategy for approaching the problem:** (2 if correct, 0 if irrelevant principle/ideas, 1 if have both some relevant and some irrelevant principles/ideas.)
2. **Explaining (in words) the reasoning that goes along with the equations/math you are doing.** (2 for correct explanation, 1 for mostly correct or incomplete explanations, 0 for incorrect explanations)
3. **Showing the details of your solution (equations/math)** (2 for correct equations/solution, 1 for mostly correct or incomplete equations/solution, 0 for incorrect equations/solution)
4. **Clarity of solution.** (2 if good, 1 if difficult but can be figured out, 0 if incomprehensible. If 0 here, all the others will be left blank.)

You now need to follow the same procedure as shown in this example to figure out what the electric field **E** is as a function of **x** and **t** in microwave oven and laser cavity. Although we have broken the solution up into a set of steps for you to follow, keep in mind these are just pieces and you should keep thinking how they fit together. In future homework problems we will expect you to go through all the steps together. In the example, we studied the wave equation for the position **x** of a mass as a function of time **t**.

Now we will study the wave equation for the electric field **E** as a function of **x** and **t**. We are starting with the fact that the in 1-dimension **E** is described by the equation:

$$\frac{\partial^2 E(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E(x,t)}{\partial t^2}$$

0. In part **A)** of the example, explain why $\mathbf{x = At^2}$ does not work as a solution, but $\mathbf{x = A\sin(Bt)}$ does. If you don't understand this, ask someone for help!

1. Show which of the following possible functional forms work or do not work as solutions to this differential equation (explaining how you know the solution works or not):

$\mathbf{E(x,\ t) = A\cos(Bt),\ E(x,\ t) = Ax^2t^2,\ E(x,\ t) = A\cos(Bx+Ct).}$

For the one(s) that will work as a solution, figure out what constraints, if any, there are on the values of the constants A, B, and C.

2. At any fixed point along x, the function **Acos(Bx+Ct)** increases and decreases as a function of time. The period is how long it takes to go through one complete cycle at any given x (say x=0, for example). What is the period and frequency of the oscillation in terms of the unknown constants in the solution? (Remember always to explain your reasoning in words).

3.(1pt) A wave repeats as a function of position as well as time. At a fixed time, the wavelength is how much *x* must change to get one full cycle of the wave. What is the wavelength in terms of the unknown constants? Explain your reasoning.

4.(2pts) Sketch what this solution will look like as a function of x at $t = 0$, $t = 0.2\pi /C$, and $t = 0.4\pi /C$. Use this sketch to determine which direction this wave is traveling. Explain what is meant by the speed of the wave and how this speed is related to the constants in the solution.

5.(2 pts) Find a second functional form for $E(x,t)$ that works as a solution you want a wave traveling in the opposite direction for simplicity! Just picking Sine as in the worked example is a much harder approach to this and will give you problems in part 6). Show that your functional form works and find any constraints there may be on the unknown constants in the solution.

The general solution is then the sum of this function and whatever functions you found work in part 1. Note that you cannot assume that the amplitudes of the two functions are the same. Why not?

For working with the general solution in parts 6 and 7, you may find the following trig identities useful when simplifying your solution to the boundary conditions:

$\cos(s+t)=\cos(s)\cos(t) - \sin(s)\sin(t)$

$\sin(s+t)=\sin(s)\cos(t) + \cos(s)\sin(t)$

$\cos(s-t)=\cos(s)\cos(t) + \sin(s)\sin(t)$

$\sin(s-t)=\sin(s)\cos(t) - \cos(s)\sin(t)$

6.(3 pts) Now we will consider the effect of the boundary conditions on the oscillating electric field in a 0.59 m wide microwave oven where the metal walls require that the electric field is always zero at the position of the wall. So $E(x=0)=0$ and $E(x=0.59\ m)=0$ for all values of t. What values can the constants in the general solution have and still satisfy these boundary conditions? What constraints does this place on the possible wavelengths in the x direction that can be inside the microwave oven? Sketch out what the field in the oven will look like for a few possible wavelengths that would work. (If you need a hint, this problem is mathematically quite similar to the problem of a wave on a violin string, with the boundary condition being that the string is held so it cannot move at either end. The solutions to the respective differential equations have corresponding similarities. You should end up with a standing wave.

7.(2pts) Microwave ovens operate at a frequency of very close to 2.54 GHz. Use this to determine the wavelength of your wave. Make a sketch, with explanation and scale showing what the electric field will look like as a function of space at several different times (e.g. $t=0$, $t=0.25\pi$, $t=0.5\ \pi$, $t=0.75\ \pi$, and $t=\pi$) in the oven. Is your picture consistent with your boundary conditions? What on your sketch represents the wavelength? The period? In a microwave oven the oscillating electric field causes the molecules in the food to oscillate back and forth and this motion is turned into heat, thereby heating up the food. How does your sketch of the field explain why there are particular locations in a microwave oven where the food does not heat as well as at other locations?

8.(1 pt) Laser light shows usually use an argon-ion laser. This type of laser produces a green light with a single wavelength of 541.5 nm. The laser beam comes out of a tube that is about 2 m long and has mirrors at each end where the electric field is forced to be zero, similar to the boundary conditions of the walls of the microwave oven. The field between the mirrors is very strong, and the laser beam that comes out is just a small part that leaks through one of the mirrors.

Describe what the electric field looks like along the axis of the laser tube. About how many half wavelengths fit between the mirrors?