Math 307: homework problems 5

1. Review of complex numbers:

   (a) Show that $|zw| = |z||w|$ for any complex numbers z and w.

   (b) Show that $\overline{zw} = \bar{z}\bar{w}$ for any complex numbers z and w.

   (c) Show that $\bar{z}z = |z|^2$ for every complex number z.

2. Calculate the inner products and norms for the following:

   (a) the real vectors $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 5 \\ -1 \end{bmatrix}$;

   (b) the complex vectors $\begin{bmatrix} 1+i \\ 3-i \\ 2+2i \\ 6-3i \end{bmatrix}$ and $\begin{bmatrix} 2-2i \\ 4+3i \\ 6-i \\ 1 \end{bmatrix}$;

   (c) the functions $x-1$ and $\cos x$ on the interval $[-\pi, \pi]$;

   (d) the functions e^{3ix} and e^{-ix} on the interval $[0, 2\pi]$.

3. Using the definition of the inner product for real vectors $\mathbf{x}$ and $\mathbf{y}$ show that $\langle \mathbf{x}, A\mathbf{y} \rangle = \langle A^T\mathbf{x}, \mathbf{y} \rangle$ for a matrix A with real entries. Similarly, using the definition of the inner product for complex vectors $\mathbf{x}$ and $\mathbf{y}$ show that $\langle \mathbf{x}, A\mathbf{y} \rangle = \langle \bar{A}^T\mathbf{x}, \mathbf{y} \rangle$ for a matrix A with complex entries.

4. Use the Cauchy-Schwarz inequality for real vectors to show

   $$\|\mathbf{x} + \mathbf{y}\|^2 \leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2$$

   Under what circumstances is the inequality an equality?

5. Compute the matrix P for the projection onto the line spanned by $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ in $\mathbb{R}^4$. Compute the matrix Q for the projection onto the (hyper-)plane orthogonal to $\mathbf{a}$.

6. Compute the matrix P for the projection onto the plane spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \\ -3 \end{bmatrix}$. (Careful: these vectors are not linearly independent so A^TA is not invertible. This means you can't use the formula $P = A(A^TA)^{-1}A^T$ directly for the matrix A containing all the vectors above as columns.)

7. Using least squares, find the best quadratic fit for the points $(1, 5)$, $(2, 3)$, $(3, 3)$, $(4, 4)$, $(5, 5)$. To do this, write down the system of linear equations for a, b and c that expresses the condition that $p(x) = ax^2 + bx + c$ passes through these points. These equations have no solution, but you can find the least square solution. Plot the result together with the points.

8. Using MATLAB/Octave, find an orthonormal set of vectors $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$ with the same span as $\begin{bmatrix}1\\1\\2\\0\\0\\0\end{bmatrix}$ $\begin{bmatrix}1\\0\\1\\0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\\1\\1\\0\end{bmatrix}$

9. Do the following computational experiment. First start with a random symmetric 10×10 matrix A (for example `B=rand(10,10); A=B'*B;` will produce such a matrix) and compute its QR factorization. Call the factors Q_1 and R_1. Now multiply Q_1 and R_1 in the "wrong" order to obtain $A_2 = R_1 Q_1$ and compute the QR factorization of the resulting matrix A_2. Repeat this step to obtain a sequence of matrices Q_k, R_k and A_k. Do these sequences converge? If so can you identify the limit? (Hint: `eig(C)` computes the eigenvalues of `C`).