Math 307: homework problems 4

1. Let D be the incidence matrix in the example done in lectures and in the online notes. Using MATLAB/Octave (or otherwise) compute $\mathbf{rref}(D)$ and find the bases for $N(D)$, $R(D)$ and $R(D^T)$. Find a basis for $N(D^T)$ by computing $\mathbf{rref}(D^T)$. Verify that every loop vector is a linear combination of vectors in this basis.

2. Draw the graph corresponding to the incidence matrix

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}.$$

3. How many disconnected components does the graph whose incidence matrix is in the file `hmkgraph.m` have (provided on the website)?

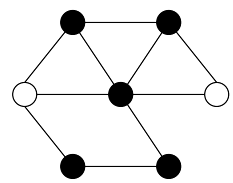
4. Write a MATLAB/Octave function `resistance(L,n,m)` that takes as input the Laplacian matrix L for a circuit and the position of two nodes n and m and returns the effective resistance between those two nodes. Here is a template for the function (provided on the website in `resistance.m`). Edit this file, replacing the stars `*****` with your code. Hand in a list of the changes that you made.

```
function r=resistance(L,n,m)

% if n=m then return with r=0;
if(n==m)
r=0;
return;
end

% if m < n the swap n and m
if(m < n)
temp=m; m=n; n=temp;
end

%find the size of the matrix L
*****

%swap the nth and mth nodes to positions 1 and 2 in L
*****

%compute the submatrices A B C and the voltage-to current map DN
*****
```

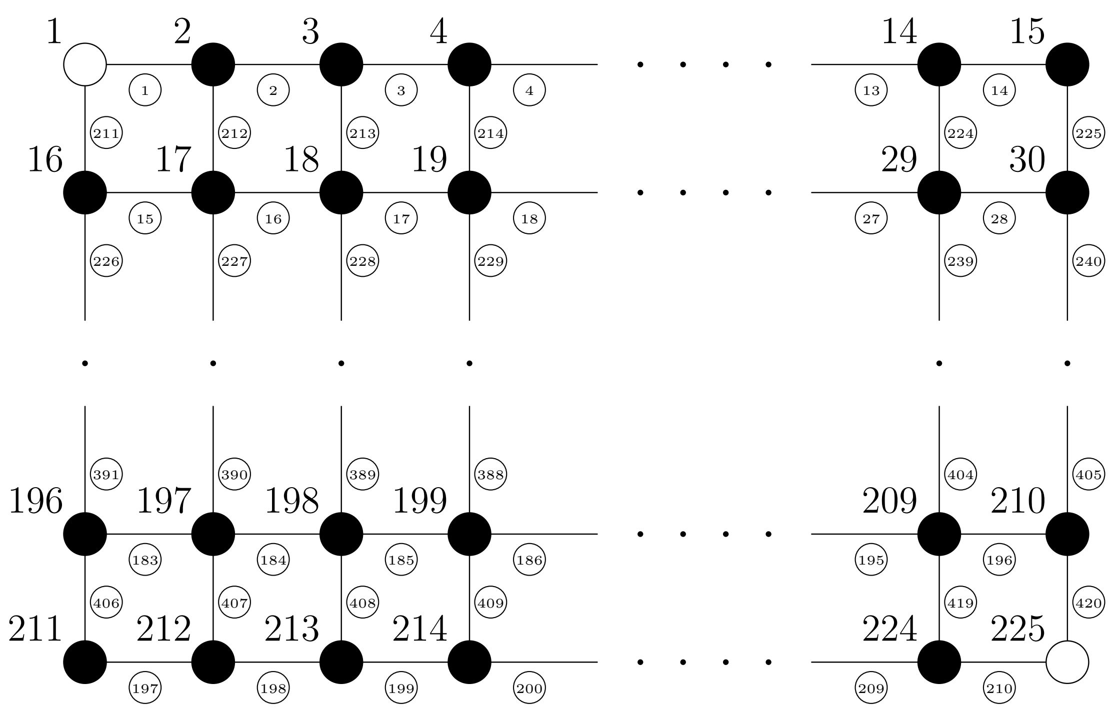
```
%the effective resistance is the reciprocal of the 1 1 entry of DN
r = 1/DN(1,1);
end
```

When you have finished, you can use this function to compute resistances.

5. Using the function you wrote in the last question, compute the effective resistance of the following network between the indicated nodes. Assume that all resistances have value $R_i = 1$.



6. (*Resistor network analogy to lightning*) Air is generally a very poor electrical conductor. Its ability to conduct depends on such factors as humidity and density, variations of which can be considered as essentially random over the depth of the troposphere. Nevertheless it can conduct, as observed in lightning strikes. This question investigates the nature of lightning, using a simple model of the electrical resistance of the atmosphere as the resistor network shown below, with resistances randomly assigned to each edge.



Consider resistances of the form $R_{\textcircled{j}} = \exp(\alpha p_{\textcircled{j}})$, where α is a parameter that we can vary and $p_{\textcircled{j}}$ is a (uniformly distributed) random number between 0 and 1 assigned to edge $\textcircled{j}$. For a resistor network of 15×15 nodes, we apply a voltage of 1 at the first node (representing a point high in the atmosphere) and 0 at the last node (representing a point on the ground). How does the path that the resulting current follows change as α is increased from 0 to 30? Write a couple of short sentences describing how you found the current through each edge. Also hand in either the list of MATLAB/Octave commands that you used to solve the problem, or a series of plots (or hand-drawn sketches of the plots if you have trouble printing) showing how the paths that the current takes change as α increases (indicate larger magnitude currents with thicker lines — see below).

For small values of α the directed graph with weighted edges that represents the resistor network is said to have *weak disorder* (essentially the weights assigned to all edges are similar) whereas for large α it is said to have *strong disorder* (essentially there is a wide range of weights). By comparing your plots with lightning, would you say that the resistance of the atmosphere has weak or strong disorder?

Hints:

- *The creation of the incidence matrix can be sped up considerably using* `for` *loops in MATLAB/Octave. These work as follows: say we want to create a vector p with 15 elements and $p_i = 3i$ for each i, then we can use*

```
for i=1:15
  p(i) = 3*i;
end
```

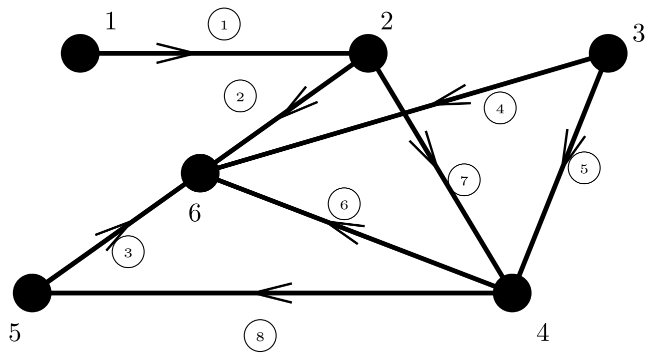
- *If you number the nodes and edges in the way suggested in the figure above (first number the horizontal edges, starting at the top left and then the vertical edges again starting at the top left) and let* `I` *be the vector of currents through the edges, then the following MATLAB/Octave commands will plot the current through each edge (with thicker lines representing larger currents):*

```
hold off
Imax = max(max(abs(I)));
for i=1:n
  for j=1:n-1
    x(1) = i;
    x(2) = i;
    y(1) = j;
    y(2) = j+1;
    plot(y,n-x,'LineWidth',round(20*abs(I((n-1)*(i-1)+j))/Imax))
    hold on
  end
end
for i=1:n-1
  for j=1:n
    x(1) = i;
    x(2) = i+1;
    y(1) = j;
    y(2) = j;
    plot(y,n-x,'LineWidth',round(20*abs(I((n-1)*n+n*(i-1)+j))/Imax))
    hold on
  end
end
axis equal
```

These commands are contained in the file `lightning_plot.m` *on the website. You can either add your commands before the plotting commands, or type these commands in at the prompt after you have found the vector* `I`.

7. The ideas we have used for resistor networks can also be applied to the flow of fluid through networks, for example air flow through connected tunnels or the flow of oil through sandstone (the paths between

pores in the sandstone correspond to edges and pores correspond to nodes). Consider the following network of tubes:



Let $Q_{\textcircled{j}}$ be the volume flux of fluid through tube $\textcircled{j}$ and q_j be the total volume flux of fluid injected/extracted from the network of tubes at node j. Let p_j be the pressure applied at each node and $P_{\textcircled{j}}$ be the pressure difference between the ends of tube $\textcircled{j}$. Let $R_{\textcircled{j}}$ be the resistance of the tube (it depends on the geometry of the tube).

The physical laws governing (slow) flow of an incompressible fluid through this network of tubes are similar to the electrical circuit rules of a resistor network:

(a) The Hagen–Poiseuille law states that $R_{\textcircled{j}} = \dfrac{P_{\textcircled{j}}}{Q_{\textcircled{j}}}$

(b) There can be no pressure difference around a closed loop.

(c) The total volume flux coming in to a node must equal the total volume flux coming out.

For the network of tubes sketched above, find the incidence matrix D. Find bases for and give the meaning of $R(D)$, $N(D)$, $R(D^T)$, $N(D^T)$.