Math 307: homework problems 3

1. Are the vectors $\begin{bmatrix}1\\2\\1\\2\\1\end{bmatrix}$, $\begin{bmatrix}1\\0\\-2\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\-1\\3\\-2\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\-2\\0\\1\end{bmatrix}$, $\begin{bmatrix}0\\4\\-9\\7\\3\end{bmatrix}$ linearly independent?

2. If $\mathbf{rref}(A) = \begin{bmatrix}1 & a & 0 & b & d & 0\\0 & 0 & 1 & c & e & 0\\0 & 0 & 0 & 0 & 0 & 1\\0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$ find a basis for $N(A)$ and $R(A^T)$.

3. Explain why the first three rows of the matrix in Q2 are linearly independent.

4. Find the rank $\mathbf{r}(A)$ and bases for $N(A)$, $R(A)$, $N(A^T)$ $R(A^T)$ when $A = \begin{bmatrix}1 & 2 & 0 & 2 & 1 & 0\\1 & 2 & 1 & 5 & 3 & 0\\1 & 2 & 1 & 5 & 3 & 1\\1 & 2 & 1 & 5 & 3 & 1\end{bmatrix}$.

5. Show that if $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linearly independent and E is invertible then $E\mathbf{u}_1, E\mathbf{u}_2, \ldots, E\mathbf{u}_k$ are also linearly independent. Is this still true if E is not invertible?

6. We saw in class that the set of polynomials of degree at most n form a subspace (of functions). If we take $n = 2$, then a general quadratic polynomial can be written as $p(x) = c_0 + c_1 x + c_2 x^2$. We can represent this polynomial by the vector $\mathbf{p} = \begin{bmatrix}c_0\\c_1\\c_2\end{bmatrix}$. The derivative of $p(x)$ is $q(x) = c_1 + 2c_2 x$, which may be represented by the vector $\mathbf{q} = \begin{bmatrix}c_1\\2c_2\\0\end{bmatrix}$.

   Show that the matrix
   $$D_2 = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 2\\0 & 0 & 0\end{bmatrix}$$
   satisfies $D_2\mathbf{p} = \mathbf{q}$ (and so D_2 represents the derivative).

   If we change the value of n, then the dimension increases. Find the matrix D_n representing the derivative for degree n polynomials.

   What is the nullspace of D_n?

   What is the range of D_n?

   Show that there is no solution to $(D_n - I)\mathbf{p} = \mathbf{0}$ except $\mathbf{p} = \mathbf{0}$. (It is easiest to do this problem by reverting to a differential equation and using properties of the derivative and polynomials of degree n.)

7. (Bonus) We showed above that the derivative acting on polynomials of degree n behaves much like a matrix acting on vectors. Here we extend some of those properties to the set of smooth functions (functions for which all derivatives exist) rather than just polynomials. We can now no longer represent the function directly as a vector or derivatives as matrices.

   We represent the derivative by D, so that $Df(x)$ is the derivative of $f(x)$ [you've previously seen df/dx or $f'(x)$ as other ways to write this].

   Find the nullspace of D.

   Find the nullspace of D^n.

   Show that e^{rx} is in the nullspace of $D - rI$, where I is the identity: $If(x) = f(x)$.

   Find the nullspace of $(D - rI)^2$. (*Hint: use integrating factors.*)

   Find the nullspace of $(D - rI)^n$.

8. (Optional) The differential equation $f''(x) - (r_1 + r_2)f'(x) + r_1 r_2 f(x) = 0$ may be written as

$$[D^2 - (r_1 + r_2)D + r_1 r_2 I]f(x) = 0$$

   We may also write this as $P(D)f(x) = 0$ where $P(D) = D^2 - (r_1 + r_2)D + r_1 r_2 I$. It is possible to factor $P(D)$ into either $(D - r_1)(D - r_2)$ or $(D - r_2)(D - r_1)$.

   Show that the nullspace of $D - r_1$ and the nullspace of $D - r_2$ are in the nullspace of $P(D)$ and hence find the general solution to the differential equation.

   Show that the solution to

$$P(D)f(x) = e^{\alpha x}$$

   is in the nullspace of $(D - \alpha I)P(D)$.

   Find the solution to

$$f''(x) - 5f'(x) + 6f(x) = e^x$$

   with $f(0) = 1, \;\; f'(0) = 2$.

   Find the solution to

$$f''(x) - 5f'(x) + 6f(x) = e^{2x}$$

   with the same initial conditions.