Math 307: homework problems 1

1. Use Gaussian elimination to find the solution(s) to $A\mathbf{x} = \mathbf{b}$ where

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 2 & -3 & 4 \\ 5 & 6 & 7 & 8 \\ -5 & 6 & -7 & 8 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \text{(b)}\quad A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

2. Use MATLAB/Octave to find the solution(s) to $A\mathbf{x} = \mathbf{b}$ where

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \text{(b)}\quad A = \begin{bmatrix} 1 & 0 & 3 & 2 & -4 \\ 2 & 1 & 6 & 5 & 0 \\ -1 & 1 & -3 & -1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 7 \\ -5 \end{bmatrix}.$$

3. Show that for any square matrix A, $\|A\|_{HS}^2 = \mathrm{tr}(A^T A)$.

4. Guess whether each of the following statements about $n \times n$ matrices A is true or false by testing them on a few random matrices. Bonus: prove that your guess is correct.
   (a) $\|A^2\| = \|A\|^2$
   (b) $\|A^2\| \le \|A\|^2$
   (c) $\|A^T A\| = \|A\|^2$
   (d) $\|A\| \le \|A\|_{HS}$
   (e) $\mathrm{cond}(A) = \mathrm{cond}(A^{-1})$
   (f) $\mathrm{cond}(A) \ge 1$

5. In "single precision" computer calculations, we cannot trust more than approximately the first 7 significant figures. Assuming that the relative error in the right-hand-side of the matrix equation $A\mathbf{x} = \mathbf{b}$ is $\|\Delta\mathbf{b}\|/\|\mathbf{b}\| = 1.1921 \times 10^{-7}$, give an upper bound on the relative error $\|\Delta\mathbf{x}\|/\|\mathbf{x}\|$ in the solution of the equation for the following matrices A:

$$\text{(a)}\quad A = \begin{bmatrix} 0 & 2 & 4 \\ 3 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \qquad \text{(b)}\quad A = \begin{bmatrix} 4 & 1.99 \\ 2.01 & 1 \end{bmatrix}$$

   Interpret these bounds to say how many significant figures we can trust in the solution.

6. Compute the determinant of a 4×4 Vandermonde matrix. Bonus: show that the general formula for the determinant of a Vandermonde matrix is correct.

7. Let V_n be the Vandermonde matrix for n equally spaced points between 0 and 1. Do you think the condition number of V_n is increasing exponentially in n? To make an informed guess, make a plot of $\log(\mathrm{cond}(V_n))$ against n.

8. Plot the Lagrange interpolating function through the points $(1, 2.3)$, $(2, 5)$, $(2.4, 9)$, $(2.5, 5)$, $(3, 0)$ and $(5, -1)$.