Chapter II

Bases and Dimensions

II.1 Basis and dimension

II.1.1 Linear dependence and independence

A *linear combination* of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a vector of the form

$$\sum_{i=1}^{k} c_i \mathbf{v}_i = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

for some choice of numbers $c_1, c_2, \ldots, c_k$.

The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are called *linearly dependent* if there exist numbers $c_1, c_2, \ldots, c_k$ that are not all zero, such that the linear combination $\sum_{i=1}^{k} c_i \mathbf{v}_i = \mathbf{0}$

On the other hand, the vectors are called *linearly independent* if the only linear combination of the vectors equalling zero has every $c_i = 0$. In other words

$$\sum_{i=1}^{k} c_i \mathbf{v}_i = \mathbf{0} \quad \text{implies} \quad c_1 = c_2 = \cdots = c_k = 0$$

For example, the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}7\\1\\7\end{bmatrix}$ are linearly dependent because

$$1\begin{bmatrix}1\\1\\1\end{bmatrix} + 6\begin{bmatrix}1\\0\\1\end{bmatrix} - 1\begin{bmatrix}7\\1\\7\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent, then at least one of the $\mathbf{v}_i$'s can be written as a linear combination of the others. To see this suppose that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

with not all of the c_i's zero. Then we can solve for any of the $\mathbf{v}_i$'s whose coefficient c_i is not zero. For instance, if c_1 is not zero we can write

$$\mathbf{v}_1 = -(c_2/c_1)\mathbf{v}_2 - (c_3/c_1)\mathbf{v}_3 - \cdots - (c_k/c_1)\mathbf{v}_k$$

This means any linear combination we can make with the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ can be achieved without using $\mathbf{v}_1$, since we can simply replace the occurrence of $\mathbf{v}_1$ with the expression on the right.

Sometimes it helps to have a geometrical picture. In three dimensional space $\mathbb{R}^3$, three vectors are linearly dependent if they lie in the same plane.

The columns of the matrix $\begin{bmatrix} 1 & * & * \\ 0 & 2 & * \\ 0 & 0 & 3 \end{bmatrix}$ are linearly independent. Here $*$ denotes an arbitrary entry. To see this suppose that

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} * \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} * \\ * \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Then, equating the bottom entries we find $3c_3 = 0$ so $c_3 = 0$. But once we know $c_3 = 0$ then the equation reads

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} * \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which implies that $c_2 = 0$ too, and similarly $c_1 = 0$

Similarly, for a matrix in echelon form (even if, as in the example below, it is not completely reduced), the pivot columns are linearly independent. For example the first, second and fifth columns in the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 5 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

are independent. For a matrix in reduced row echelon form, like

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

the pivot columns are standard basis vectors (see below), which are obviously independent.

These examples motivate the following matrix formulation. Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$, put them in the columns of a matrix A so that

$$A = \big[[c|c|c|c] \mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_k \big].$$

If we put the coefficients $c_1, c_2, \ldots, c_k$ into a vector

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

then

$$A\mathbf{c} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

is the linear combination of the columns $\mathbf{v}_1, \ldots, \mathbf{v}_k$ with coefficients c_i.

The vectors are linearly dependent if there is a non-zero solution $\mathbf{c}$ to the homogeneous equation

$$A\mathbf{c} = \mathbf{0}$$

On the other hand, if the only solution to the homogeneous equation is $\mathbf{c} = \mathbf{0}$ then the columns $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

To compute whether a given collection of vectors is dependent or independent we can place them in the columns of a matrix A and reduce to echelon form. If the echelon form has only pivot columns, then there are no non-zero solutions to $A\mathbf{c} = \mathbf{0}$ and the vectors are independent. On the other hand, if the echelon form has some non-pivot columns, then the equation $A\mathbf{c} = \mathbf{0}$ has some non-zero solutions and so the vectors are dependent.

Let's try this with the vectors in the example above in MATLAB/Octave.

```
>V1=[1 1 1]';
>V2=[1 0 1]';
>V3=[7 1 7]';
>A=[V1 V2 V3]

A =

   1   1   7
   1   0   1
   1   1   7

>rref(A)

ans =

   1   0   1
   0   1   6
   0   0   0
```

Since the third column is a non-pivot column, the vectors are linearly dependent.

II.1.2 Subspaces

A collections of vectors V is called a *subspace* if linear combinations of vectors from V stay inside V. In other words, if $\mathbf{v}_1$ and $\mathbf{v}_2$ lie in V then so does $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

In three dimensional space, examples of subspaces are lines and planes through the origin. If we add or scalar multiply two vectors lying on the same line (or plane) the resulting vector remains on the same line (or plane). Additional subspaces are the trivial subspace, containing the single vector $\mathbf{0}$, as well as the whole space itself.

Here is another example of a subspace. The set of $n \times n$ matrices can be thought of as an n^2 dimensional vector space. Within this vector space, the set of symmetric matrices (satisfying $A^T = A$) is a subspace. To see this, suppose A_1 and A_2 are symmetric. Then, using the linearity property of the transpose, we see that

$$(c_1A_1 + c_2A_2)^T = c_1A_1^T + c_2A_2^T = c_1A_1 + c_2A_2$$

which shows that $c_1A_1 + c_2A_2$ is symmetric too.

Given a collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ we may form a subspace of all possible linear combinations. This is a subspace is called $\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ or the space spanned by the v_i's. It is a subspace because if we start with any two elements of $\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$, say $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ and $d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_k\mathbf{v}_k$ then a linear combination of these linear combinations is again a linear combination since

$$\begin{aligned} s_1(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) + s_2(d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_k\mathbf{v}_k) = \\ (s_1c_1 + s_2d_1)\mathbf{v}_1 + (s_1c_2 + s_2d_2)\mathbf{v}_2 + \cdots + (s_1c_k + s_2d_k)\mathbf{v}_k \end{aligned}$$

For example the span of the three vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$ is the whole three dimensional space, because every vector is a linear combination of these. The span of the four vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is the same.

II.1.3 Nullspace $N(A)$ and Range $R(A)$

There are two important subspaces associated to any matrix. Let A be an $n \times m$ matrix. If $\mathbf{x}$ is m dimensional, then $A\mathbf{x}$ makes sense and is a vector in n dimensional space.

The first subspace associated to A is the *nullspace* (or *kernel*) of A denoted $N(A)$ (or $\mathrm{Ker}(A)$). It is defined as all vectors $\mathbf{x}$ solving the homogeneous equation for A, that is

$$N(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$$

This is a subspace because if $A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$ then

$$A(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1A\mathbf{x}_1 + c_2A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

The nullspace is a subspace of m dimensional space $\mathbb{R}^m$.

The second subspace is the range (or column space) of A denoted $R(A)$ (or $C(A)$). It is defined as all vectors of the form $A\mathbf{x}$ for some $\mathbf{x}$. From our discussion above, we see that $R(A)$ is the set of all possible linear combination (or the span) of its columns. This explains the name "column space". The range is a subspace of n dimensional space $\mathbb{R}^n$.

II.1.4 Basis

A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ contained in a subspace V is called a *basis* for that subspace if

1. $\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = V$, and
2. $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

Condition (1) says that any vector in V can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Condition (2) says that there is exactly one way of doing this. Here is the argument. Suppose there are two ways of writing the same vector $\mathbf{v} \in V$ as a linear combination:

$$\begin{aligned} \mathbf{v} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \\ \mathbf{v} &= d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_k\mathbf{v}_k \end{aligned}$$

Then by subtracting these equations, we obtain

$$\mathbf{0} = (c_1 - d_1)\mathbf{v}_1 + (c_2 - d_2)\mathbf{v}_2 + \cdots + (c_k - d_k)\mathbf{v}_k$$

Linear independence now says that every coefficient in this sum must be zero. This implies $c_1 = d_1$, $c_2 = d_2$ $\ldots$ $c_k = d_k$.

Example: $\mathbb{R}^n$ has the standard basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \\ \vdots \end{bmatrix} \quad \mathbf{e}_1 = \begin{bmatrix} 0 \\ 1 \\ \\ \vdots \end{bmatrix} \quad \cdots$$

Another basis for $\mathbb{R}^2$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. To see this, notice that saying that any vector $\mathbf{y}$ can be written in a unique way as $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the same as saying that the equation

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{x}$$

always has a unique solution. This is true.

It is intuitively clear that, say, a plane in three dimensions will always have a basis of two vectors. Here is an argument that shows that any two bases for a subspace V will always have the same number of elements. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{w}_1, \ldots, \mathbf{w}_m$ be two bases for a subspace

V. Let's try to show that n must be the same as m. Since the $\mathbf{v}_i$'s span V we can write each $\mathbf{w}_i$ as a linear combination of $\mathbf{v}_i$'s. We write

$$\mathbf{w}_j = \sum_{i=1}^{n} a_{i,j}\mathbf{v}_i$$

for each $j = 1, \ldots, m$. Let's put all the coefficients into an $n \times m$ matrix $A = [a_{i,j}]$.

Now suppose that $m > n$. Then A has more columns than rows. So its echelon form must have some non-pivot columns which implies that there must be some non-zero solution to $A\mathbf{c} = \mathbf{0}$. Let $\mathbf{c} \neq \mathbf{0}$ be such a solution. The equation $A\mathbf{c} = \mathbf{0}$ can be written out as the system of equations

$$\sum_{j=1}^{m} a_{i,j}c_j = 0$$

for $i = 1, \ldots, n$. Now we compute

$$\begin{aligned}
\sum_{j=1}^{m} c_j\mathbf{w}_j &= \sum_{j=1}^{m} c_j \left(\sum_{i=1}^{n} a_{i,j}\mathbf{v}_i\right) \\
&= \sum_{j=1}^{m}\sum_{i=1}^{n} c_j a_{i,j}\mathbf{v}_i \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} a_{i,j}c_j\mathbf{v}_i \\
&= \sum_{i=1}^{n}\left(\sum_{j=1}^{m} a_{i,j}c_j\right)\mathbf{v}_i \\
&= \mathbf{0}
\end{aligned}$$

Since the c_j's are not all zeros this contradicts the linear independence of the $\mathbf{w}_j$' s and is therefore impossible. So it can't be true that $m > n$. We can eliminate the possibility that $n > m$ in the same way by exchanging the roles of $\mathbf{v}_i$ and $\mathbf{w}_j$ in the argument. The only remaining possibility is that $n = m$.

The *dimension* of a subspace V is defined to be the number of elements in any basis for V.

II.1.5 Finding basis and dimension of $N(A)$

Example: Let

$$A = \begin{bmatrix} 1 & 3 & 3 & 10 \\ 2 & 6 & -1 & -1 \\ 1 & 3 & 1 & 4 \end{bmatrix}.$$

To calculate a basis for the nullspace $N(A)$ and determine its dimension we need to find the solutions to $A\mathbf{x} = \mathbf{0}$. To do this we first reduce A to reduced row echelon form U and solve $U\mathbf{x} = \mathbf{0}$ instead, since this has the same solutions as the original equation.

```
>A=[1 3 3 10;2 6 -1 -1;1 3 1 4];
>rref(A)

ans =

   1   3   0   1
   0   0   1   3
   0   0   0   0
```

This means that $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is in $N(A)$ if

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We now divide the variables into basic variables, corresponding to pivot columns, and free variables, corresponding to non-pivot columns. In this example the basic variables are x_1 and x_3 while the free variables are x_2 and x_4. The free variables are the parameters in the solution. We can solve for the basic variables in terms of the free ones, giving $x_3 = -3x_4$ and $x_1 = -3x_2 - x_4$. This leads to

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 - x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

The vectors $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}$ span the nullspace since every element of $N(A)$ is a linear combination of them. They are also linearly independent because if the linear combination on the right of the equation above is zero, then by looking at the second entry of the vector (corresponding to the first free variable) we find $x_2 = 0$ and looking at the last entry (corresponding to the second free variable) we find $x_4 = 0$. So both coefficients must be zero.

To find a basis for $N(A)$ in general we first compute $U = \mathrm{rref}(A)$ and determine which variables are basic and which are free. For each free variable we form a vector as follows. First put a 1 in the position corresponding to that free variable and a zero in every other free variable position. Then fill in the rest of the vector in such a way that $U\mathbf{x} = \mathbf{0}$. (This is easy to do!) The set all such vectors - one for each free variable - is a basis for $N(A)$.

II.1.6 The matrix version of Gaussian elimination

How are a matrix A and its reduced row echelon form $U = \text{rref}(A)$ related? If A and U are $n \times m$ matrices, then there exists an invertible $n \times n$ matrix such that

$$A = EU \quad E^{-1}A = U$$

This immediately explains why the $N(A) = N(U)$, because if $A\mathbf{x} = \mathbf{0}$ then $U\mathbf{x} = E^{-1}A\mathbf{x} = \mathbf{0}$ and conversely if $A\mathbf{x} = \mathbf{0}$ then $U\mathbf{x} = EA\mathbf{x} = \mathbf{0}$.

What is this matrix E? It can be thought of as a matrix record of the Gaussian elimination steps taken to reduce A to U. It turns out performing an elementary row operation is the same as multiplying on the left by an invertible square matrix. This invertible square matrix, called an elementary matrix, is obtained by doing the row operation in question to the identity matrix.

Suppose we start with the matrix

```
>A=[1 3 3 10;2 6 -1 -1;1 3 1 4]

A =

    1    3    3   10
    2    6   -1   -1
    1    3    1    4
```

The first elementary row operation that we want to do is to subtract twice the first row from the second row. Let do this to the 3×3 identity matrix I (obtained with `eye(3)` in MATLAB/Octave) and call the result `E1`

```
>E1 = eye(3)

E1 =

   1   0   0
   0   1   0
   0   0   1

>E1(2,:) = E1(2,:)-2*E1(1,:)

E1 =

   1   0   0
  -2   1   0
   0   0   1
```

Now if we multiply `E1` and `A` we obtain

```
>E1*A

ans =

    1    3    3   10
    0    0   -7  -21
    1    3    1    4
```

which is the result of doing that elementary row operation to A. Let's do one more step. The second row operation we want to do is to subtract the first row from the third. Thus we define

```
>E2 = eye(3)

E2 =

   1   0   0
   0   1   0
   0   0   1


>E2(3,:) = E2(3,:)-E2(1,:)

E2 =

   1   0   0
   0   1   0
  -1   0   1
```

and we find

```
>E2*E1*A

ans =

    1    3    3   10
    0    0   -7  -21
    0    0   -2   -6
```

which is one step further along in the Gaussian elimination process. Continuing in this way until we eventually arrive at U so that

$$E_k E_{k-1} \cdots E_2 E_1 A = U$$

Thus $A = EU$ with $E = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}$. For the example above it turns out that

$$E = \begin{bmatrix} 1 & 3 & -6 \\ 2 & -1 & -18 \\ 1 & 1 & -9 \end{bmatrix}$$

which we can check:

```
>A=[1 3 3 10;2 6 -1 -1;1 3 1 4]

A =

    1    3    3   10
    2    6   -1   -1
    1    3    1    4

>U=rref(A)

U =

   1   3   0   1
   0   0   1   3
   0   0   0   0

>E=[1 3 -6; 2 -1 -18; 1 1 -9];
>E*U

ans =

    1    3    3   10
    2    6   -1   -1
    1    3    1    4
```

If we do a partial elimination then at each step we can write $A = E'U'$ where U' is the resulting matrix at the point we stopped, and E' is obtained from the Gaussian elimination step up to that point. A common place to stop is when U' is in echelon form, but the entries above the pivots have not been set to zero. If we can achieve this without doing any row swaps along the way then E' turns out to be lower triangular matrix. Since U' is upper triangular, this is called the LU decomposition of A.

II.1.7 A basis for $R(A)$

The ranges or column spaces $R(A)$ and $R(U)$ are not the same in general, but they are related. In fact, the vectors in $R(A)$ are exactly all the vectors in $R(U)$ multiplied by E, where E is the invertible matrix in the equation $A = EU$. We can write this relationship as

$$R(A) = ER(U)$$

To see this notice that if $\mathbf{x} \in R(U)$, that is, $\mathbf{x} = U\mathbf{y}$ for some $\mathbf{y}$ then $E\mathbf{x} = EU\mathbf{y} = A\mathbf{y}$ is in $R(A)$. Conversely if $\mathbf{x} \in R(A)$, that is, $\mathbf{x} = A\mathbf{y}$ for some $\mathbf{y}$ then $\mathbf{x} = EE^{-1}A\mathbf{y} = EU\mathbf{y}$ so $\mathbf{x}$ is E times a vector in $R(U)$.

Now if we can find a basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ for $R(U)$ the vectors $E\mathbf{u}_1, E\mathbf{u}_2, \ldots, E\mathbf{u}_k$ form a basis for $R(A)$. (Homework exercise)

But a basis for the column space $R(U)$ is easy to find. They are exactly the pivot columns of U. If we multiply these by E we get a basis for $R(A)$. But if

$$A = \left[\mathbf{a}_1 \middle| \mathbf{a}_2 \middle| \cdots \middle| \mathbf{a}_m\right], \quad U = \left[\mathbf{u}_1 \middle| \mathbf{u}_2 \middle| \cdots \middle| \mathbf{u}_m\right]$$

then the equation $A = EU$ can be written

$$\left[\mathbf{a}_1 \middle| \mathbf{a}_2 \middle| \cdots \middle| \mathbf{a}_m\right] = \left[E\mathbf{u}_1 \middle| E\mathbf{u}_2 \middle| \cdots \middle| E\mathbf{u}_m\right]$$

From this we see that the columns of A that correspond to pivot columns of U form a basis for $R(A)$. This implies that the dimension of $R(A)$ is the number of pivot columns in U.

II.1.8 The rank of a matrix

We define the rank of the matrix A, denoted $\mathrm{r}(A)$ to be the number of pivot columns of U. Then we have shown that for an $n \times m$ matrix A

$$\begin{aligned}\dim(R(A)) &= \mathrm{r}(A)\\ \dim(N(A)) &= m - \mathrm{r}(A)\end{aligned}$$

II.1.9 Bases for $R(A^T)$ and $N(A^T)$

Of course we could find $R(A^T)$ and $N(A^T)$ by computing the reduced row echelon form for A^T and following the steps above. But then we would miss an important relation between the dimensions of these spaces.

Let's start with the column space $R(A^T)$. The columns of A^T are the rows of A (written as column vectors instead of row vectors). So $R(A^T)$ is the row space of A.

It turns out that $R(A^T)$ and $R(U^T)$ are the same. This follows from $A = EU$. To see this take the transpose of this equation. Then $A^T = U^T E^T$. Now suppose that $\mathbf{x} \in R(A^T)$. This means that $\mathbf{x} = A^T\mathbf{y}$ for some $\mathbf{y}$. But then $\mathbf{x} = U^T E^T \mathbf{y} = U^T\mathbf{y}'$ where $\mathbf{y}' = E^T\mathbf{y}$ so $\mathbf{x} \in R(U^T)$. Similarly, if $\mathbf{x} = U^T\mathbf{y}$ for some $\mathbf{y}$ then $\mathbf{x} = U^T E^T (E^T)^{-1}\mathbf{y} = A^T(E^T)^{-1}\mathbf{y} = A^T\mathbf{y}'$ for $\mathbf{y}' = (E^T)^{-1}\mathbf{y}$. So every vector in $R(U^T)$ is also in $R(A^T)$. Here we used that E and hence E^T is invertible.

Now we know that $R(A^T) = R(U^T)$ is spanned by the columns of U^T. But since U^T is in reduced row echelon form, its non-zero columns are independent. Therefore, the non-zero columns of U^T form a basis for $R(A^T)$. There is one of these for every pivot. This leads to

$$\dim(R(A^T)) = \mathrm{r}(A) = \dim(R(A))$$

The final subspace to consider is $N(A^T)$. From our work above we know that

$$\dim(N(A^T)) = n - \dim(R(A^T)) = n - \mathrm{r}(A).$$

Finding a basis is tricker. It might be easiest to find the reduced row echelon form of A^T. But if we insist on using $A = EU$ or $A^T = U^T E^T$ we could proceed by multiplying on the right be the inverse of E^T. This gives

$$A^T(E^T)^{-1} = U^T$$

Now notice that the last $n - \mathrm{r}(A)$ columns of U^T are zero, since U is in reduced row echelon form. So the last $n - \mathrm{r}(A)$ columns of $(E^T)^{-1}$ are in the the nullspace of A^T. They also have to be independent, since $(E^T)^{-1}$ is invertible.

Thus the last $n - \mathrm{r}(A)$ of $(E^T)^{-1}$ form a basis for $N(A^T)$.

From a practical point of view, this is not so useful since we have to compute the inverse of a matrix. It might be just as easy to reduce A^T. (Actually, things are slightly better if we use the LU decomposition. The same argument shows that the last $n - \mathrm{r}(A)$ columns of $(L^T)^{-1}$ also form a basis for $N(A^T)$. But L^T is an upper triangular matrix, so its inverse is faster to compute.)

Summary: Math Concepts

- Linear dependence and independence: understand the definition
- Be able to decide if a collection of vectors are dependent
- Subspaces: what does it mean for a collection of vectors to form a subspace?
- Be able to decide if a collection of vectors is a subspace
- Basis: definition, check if a collection of vectors is a basis.

- Matrix form of Gaussian elimination, $A = EU$
- The four subspaces $N(A)$, $R(A)$, $N(A^T)$ $R(A^T)$ and how to compute bases for each one.
- The rank of the matrix and the formulas for the dimension of each of the four subspaces.

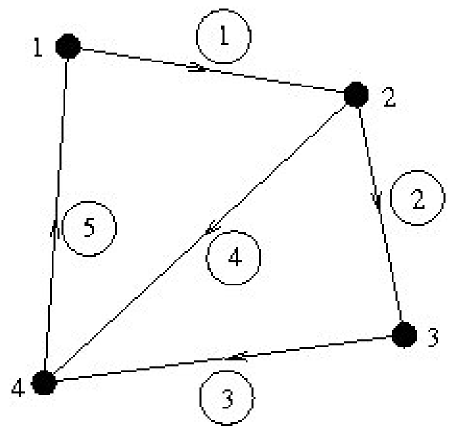
Summary: MATLAB/Octave Concepts

- `eye(n)` gives an $n \times n$ identity matrix.

II.2 Graphs and Networks

II.2.1 Directed graphs and their incidence matrix

A *directed graph* is a collection of vertices (or nodes) connected by edges with arrows. Here is a graph with 4 vertices and 5 edges.



Graphs come up in many applications. For example, the nodes could represent computers and the arrows internet connections. Or the nodes could be factories and the arrows represent movement of goods. We will mostly focus on a single interpretation where the edges represent resistors or batteries hooked up in a circuit.

The incidence matrix of a graph is an $n \times m$ matrix, where n is the number of edges and m is the number of vertices. We label the rows by the edges in the graph and the columns by the vertices. Each row of the matrix corresponds to an edge in the graph. It has a -1 in the place corresponding to the vertex where the arrow starts and a 1 in the place corresponding to the vertex where the arrow ends.

Here is the incidence matrix for the illustrated graph.

$$\begin{array}{c} \\ ① \\ ② \\ ③ \\ ④ \\ ⑤ \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

The columns of the matrix have the following interpretation. The column representing a given vertex has a $+1$ for each arrow coming in to that vertex and a -1 for each arrow leaving the vertex.

Given an incidence matrix, the corresponding graph can easily be drawn. What is the graph for

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}?$$

(Answer: a triangular loop.)

II.2.2 Nullspace and range of incidence matrix and its transpose

We now wish to give an interpretation of the fundamental subspaces associated with the incidence matrix of a graph. Let's call the matrix D. In our example D acts on vectors $\mathbf{v} \in \mathbb{R}^4$ and produces a vector $D\mathbf{v}$ in $\mathbb{R}^5$. We can think of the vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ as an assignment of a voltage to each of the nodes in the graph. Then the vector $D\mathbf{v} = \begin{bmatrix} v_2 - v_1 \\ v_3 - v_2 \\ v_4 - v_3 \\ v_4 - v_2 \\ v_1 - v_4 \end{bmatrix}$ assigns to each edge the voltage difference across that edge. The matrix D is similar to the derivative matrix when we studied finite difference approximations. It can be thought of as the derivative matrix for a graph.

II.2.3 The null space $N(D)$:

This is the set of voltages $\mathbf{v}$ for which the voltage differences in $D\mathbf{v}$ are all zero. This means that any two nodes connected by an edge will have the same voltage. In our example, this implies all the voltages are the same, so every vector in $N(D)$ is of the form $\mathbf{v} = s \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ for some s. In other words, the null space is one dimensional with basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

For a graph that has several disconnected pieces, $D\mathbf{v} = \mathbf{0}$ will force $\mathbf{v}$ to be constant on each connected component of the graph. Each connected component will contribute one basis vector to $N(D)$. This is the vector that is equal to 1 on that component and zero everywhere else. Thus $\dim(N(D))$ will be equal to the number of disconnected pieces in the graph.

II.2.4 The range $R(D)$:

The range of D consists of all vectors $\mathbf{b}$ in $\mathbb{R}^5$ that are voltage differences, i.e., $\mathbf{b} = D\mathbf{v}$ for some $\mathbf{v}$. We know that the dimension of $R(D)$ is $4 - \dim(N(D)) = 4 - 1 = 3$. So the set of voltage difference vectors must be restricted in some way. In fact a voltage difference vector will have the property that the sum of the differences around a closed loop is zero. In the example the edges ①, ④, ⑤ form a loop, so if $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$ is a voltage difference vector then $b_1 + b_4 + b_5 = 0$ We can check this directly in the example. Since $\mathbf{b} = D\mathbf{v} = \begin{bmatrix} v_2 - v_1 \\ v_3 - v_2 \\ v_4 - v_3 \\ v_4 - v_2 \\ v_1 - v_4 \end{bmatrix}$ we check that $(v_2 - v_1) + (v_4 - v_2) + (v_1 - v_4) = 0$. In the example graph there are three loops, namely ①, ④, ⑤ and ②, ③, ④ and ①, ②, ③, ⑤. The corresponding equations that the components of a vector $\mathbf{b}$ must satisfy to be in the range of D are

$$\begin{aligned} b_1 + b_4 + b_5 &= 0 \\ b_2 + b_3 - b_4 &= 0 \\ b_1 + b_2 + b_3 + b_5 &= 0 \end{aligned}$$

Notice the minus sign in the second equation corresponding to a backwards arrow. However these equations are not all independent, since the third is obtained by adding the first two. There are two independent equations that the components of $\mathbf{b}$ must satisfy. Since $R(D)$ is 3 dimensional, there can be no additional constraints.

Now we wish to find interpretations for the null space and the range of D^T. Let $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$ be a vector in $\mathbb{R}^5$ which we interpret as being an assignment of a current to each edge in the graph. Then $D^T\mathbf{y} = \begin{bmatrix} y_5 - y_1 \\ y_1 - y_2 - y_4 \\ y_2 - y_3 \\ y_3 + y_4 - y_5 \end{bmatrix}$. This vector assigns to each node the amount of current collecting at that node.

II.2.5 The null space $N(D^T)$:

This is the set of current vectors $\mathbf{y} \in \mathbb{R}^5$ which do not result in any current building up (or draining away) at any of the nodes. We know that the dimension of this space must be

$5 - \dim(R(D^T)) = 5 - \dim(R(D)) = 5 - 3 = 2$. We can guess at a basis for this space by noting that current running around a loop will not build up at any of the nodes. The loop vector $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ represents a current running around the loop ①, ④, ⑤. We can verify that this vector lies in the null space of D^T:

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The current vectors corresponding to the other two loops are $\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$. However these three vectors are not linearly independent. Any choice of two of these vectors are independent, and form a basis.

II.2.6 The range $R(D^T)$:

This is the set of vectors in $\mathbb{R}^4$ of the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = D^T\mathbf{y}$. With our interpretation these are vectors which measure how the currents in $\mathbf{y}$ are building up or draining away from each node. Since the current that is building up at one node must have come from some other nodes, it must be that

$$x_1 + x_2 + x_3 + x_4 = 0$$

In our example, this can be checked directly. This one condition in $\mathbb{R}^4$ results in a three dimensional subspace.

II.2.7 Summary and Orthogonality relations

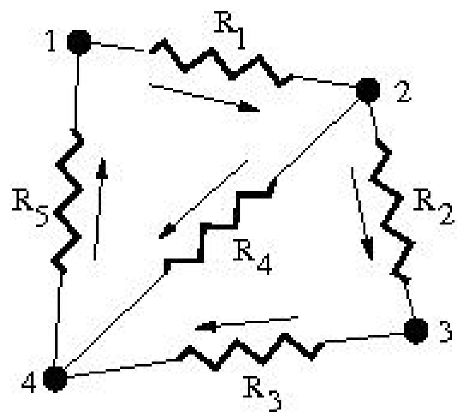
The two subspaces $R(D)$ and $N(D^T)$ are subspaces of $\mathbb{R}^5$.. The subspace $N(D^T)$ contains all linear combination of loop vectors, while $R(D)$ contains vectors whose dot product with loop vectors is zero. This means every vector in $R(D)$ is orthogonal to every vector in $N(D^T)$. We write this as $R(D) \perp N(D^T)$.

The two subspaces $N(D)$ and $R(D^T)$ are subspaces of $\mathbb{R}^4$. The subspace $N(D)$ contains constant vectors, while $R(D^T)$ contains vectors orthogonal to constant vectors. So again $N(D) \perp R(D^T)$.

It turns out that these orthogonality relations between the subspaces are valid for any matrix.

II.2.8 Resistors and the Laplacian

Now we suppose that each edge of our graph represents a resistor. This means that we associate with the ith edge a resistance R_i. Sometimes it is convenient to use conductances γ_i which are defined to be the reciprocals of the resistances, that is, $\gamma_i = 1/R_i$.



If we begin by an assignment of voltage to every node, and put these numbers in a vector $\mathbf{v} \in \mathbb{R}^4$. Then $D\mathbf{v} \in \mathbb{R}^5$ represents the vector of voltage differences for each of the edges.

Given the resistance R_i for each edge, we can now invoke Ohm's law to compute the current flowing through each edge. For each edge, Ohm's law states that

$$V_i = I_i R_i,$$

where V is the voltage drop across the edge, I_i is the current flowing through that edge and R_i is the resistance. Solving for the current we obtain

$$I_i = R_i^{-1} V_i.$$

Notice that the voltage drop V_i in this formula is exactly the ith component of the vector $D\mathbf{v}$. So if we collect all the currents in a vector $\mathbf{I}$ then Ohm's law for all the edges can be written as

$$\mathbf{I} = R^{-1} D\mathbf{v}$$

where

$$R = \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & 0 \\ 0 & 0 & R_3 & 0 & 0 \\ 0 & 0 & 0 & R_4 & 0 \\ 0 & 0 & 0 & 0 & R_5 \end{bmatrix}$$

is the diagonal matrix with the resistances on the diagonal.

Finally, if we multiply $\mathbf{I}$ by the matrix D^T the resulting vector

$$D^T\mathbf{I} = D^T R^{-1} D\mathbf{v}$$

has one entry for each node representing the total current building up or draining away at each node.

The matrix

$$L = D^T R^{-1} D$$

appearing in this formula is called the Laplacian. It is similar to the second derivative matrix that appeared when we studied finite difference approximations. Let's determine its entries.

To start we consider the case where all the resistances have the same value 1 so that $R = R^{-1} = I$. In this case $L = D^T D$. Let's start with the example graph above. Then

$$L = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

Notice that the ith diagonal entry is the total number of edges connected to the ith node. The i, j entry is -1 if the ith node is connected to the jth node, and 0 otherwise.

This pattern describes the Laplacian L for any graph. To see this, write

$$D = [\mathbf{d}_1|\mathbf{d}_2|\mathbf{d}_3|\cdots|\mathbf{d}_m]$$

Then the i, j entry of $D^T D$ is $\mathbf{d}_i^T\mathbf{d}_j$. Recall that $\mathbf{d}_i$ has an entry of -1 for every edge leaving the ith node, and a 1 for every edge coming in. So $\mathbf{d}_i^T\mathbf{d}_i$, the diagonal entries of $D^T D$, are the sum of $(\pm 1)^2$, with one term for each edge connected to the ith node. This sum gives the the total number of edges connected to the ith node. To see this in the example graph, let's consider the first node. This node has two edges connected to it and

$$\mathbf{d}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus the $1,1$ entry of the Laplacian is

$$\mathbf{d}_1^T\mathbf{d}_1 = (-1)^2 + 1^2 = 2$$

On the other hand, if $i \neq j$ then the vectors $\mathbf{d}_i$ and $\mathbf{d}_j$ have a non-zero entry in the same position only if one of the edges leaving the ith node is coming in to the jth node or vice versa. For a graph with at most one edge connecting any two nodes (we usually assume this) this means that $\mathbf{d}_i^T\mathbf{d}_j$ will equal -1 if the ith and jth nodes are connected by an edge, and zero otherwise. For example, in the graph above the the first edge leaves the first node, so that $\mathbf{d}_1$ has a -1 in the first position. This first edge comes in to the second node so $\mathbf{d}_2$ has a $+1$ in the first position. Otherwise, there is no overlap in these vectors, since no other edges touch both these nodes. Thus

$$\mathbf{d}_1^T\mathbf{d}_2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = -1$$

What happens if the resistances are not all equal to one? In this case we must replace D with $R^{-1}D$ in the calculation above. This multiplies the kth row of D with $\gamma_k = 1/R_k$. Making this change in the calculations above leads to the following prescription for calculating the entries of L. The diagonal entries are given by

$$L_{i,i} = \sum_k \gamma_k$$

Where the sum goes over all edges touching the ith node. When $i \neq j$ then

$$L_{i,j} = \begin{cases} -\gamma_k & \text{if nodes } i \text{ and } j \text{ are connected with edge } k \\ 0 & \text{if nodes } i \text{ and } j \text{ are not connected} \end{cases}$$

II.2.9 Kirchhoff's law and the null space of L

Kirchhoff's law states that currents cannot build up at any node. If $\mathbf{v}$ is the voltage vector for a circuit, then we saw that $L\mathbf{v}$ is the vector whose ith entry is the total current building up at the ith node. Thus, for an isolated circuit that is not hooked up to any batteries, Kirchhoff's law can be written as

$$L\mathbf{v} = \mathbf{0}$$

By definition, the solutions are exactly the vectors in the nullspace $N(L)$ of L. It turns out that $N(L)$ is the same as $N(D)$, which contains all constant voltage vectors. This is what we should expect. If there are no batteries connected to the circuit the voltage will be the same everywhere and no current will flow.

To see $N(L) = N(D)$ we start with a vector $\mathbf{v} \in N(D)$. Then $D\mathbf{v} = \mathbf{0}$ implies $L\mathbf{v} = D^T R^{-1} D\mathbf{v} = D^T R^{-1}\mathbf{0} = \mathbf{0}$. This show that $\mathbf{v} \in N(L)$ too, that is, $N(D) \subseteq N(L)$

To show the opposite inclusion we first note that the matrix R^{-1} can be factored into a product of invertible matrices $R^{-1} = R^{-1/2}R^{-1/2}$ where $R^{-1/2}$ is the diagonal matrix with diagonal entries $1/\sqrt{R_i}$. This is possible because each R_i is a positive number. Also, since $R^{-1/2}$ is a diagonal matrix it is equal to its transpose, that is, $R^{-1/2} = (R^{-1/2})^T$

Now suppose that $L\mathbf{v} = \mathbf{0}$. This can be written $D^T(R^{-1/2})^T R^{-1/2} D\mathbf{v} = \mathbf{0}$. Now we multiply on the left with $\mathbf{v}^T$. This gives

$$\mathbf{v}^T D^T (R^{-1/2})^T R^{-1/2} D\mathbf{v} = (R^{-1/2} D\mathbf{v})^T R^{-1/2} D\mathbf{v} = 0$$

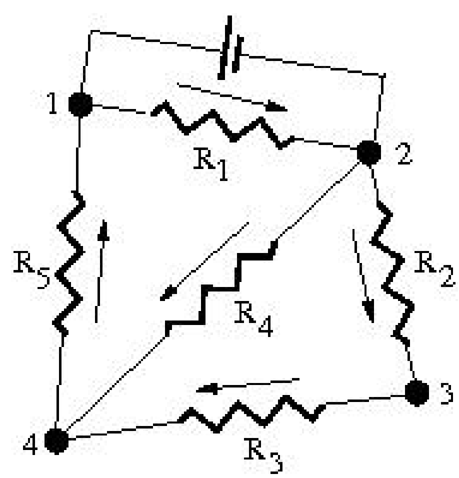
But for any vector $\mathbf{w}$, the number $\mathbf{w}^T\mathbf{w}$ is the dot product of $\mathbf{w}$ with itself which is equal to the length of $\mathbf{w}$ squared. Thus the equation above can be written

$$\|R^{-1/2} D\mathbf{v}\|^2 = 0$$

This implies that $R^{-1/2}D\mathbf{v} = \mathbf{0}$. Finally, since $R^{-1/2}$ is invertible, this yields $D\mathbf{v} = \mathbf{0}$. We have shown that any vector in $N(L)$ also is contained in $N(D)$. Thus $N(L) \subseteq N(D)$ and together with the previous inclusion this yields $N(L) = N(D)$.

II.2.10 Connecting a battery

To see more interesting behaviour in a circuit, we pick two nodes and connect them to a battery. For example, let's take our example circuit above and connect the nodes 1 and 2.



The terminals of a battery are kept at a fixed voltage. Thus the voltages v_1 and v_2 are now known say,

$$v_1 = b_1$$
$$v_2 = b_2$$

Of course, it is only voltage differences that have physical meaning, so we could set $b_1 = 0$. Then b_2 would be the voltage of the battery.

At the first and second nodes there now will be current flowing in and out from the battery. Let's call these currents I_1 and I_2. At all the other nodes the total current flowing in and out is still zero, as before.

How are the equations for the circuit modified? For simplicity let's set all the resistances $R_i = 1$. The new equations are

$$\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ 0 \\ 0 \end{bmatrix}$$

Two of the voltages v_1 and v_2 have changed their role in these equations from being unknowns to being knowns. On the other hand, the first two currents, which were originally known quantities (namely zero) are now unknowns.

To solve this system of equations we write it in block matrix form

$$\begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix}$$

where

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} v_3 \\ v_4 \end{bmatrix} \quad \mathbf{I} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Our system of equations can then be written as two 2×2 systems.

$$\begin{aligned} A\mathbf{b} + B^T\mathbf{v} &= \mathbf{I} \\ B\mathbf{b} + C\mathbf{v} &= \mathbf{0} \end{aligned}$$

We can solve the second equation for $\mathbf{v}$. Since C is invertible

$$\mathbf{v} = -C^{-1}B\mathbf{b}$$

Using this value of $\mathbf{v}$ in the first equation yields

$$\mathbf{I} = (A - B^T C^{-1} B)\mathbf{b}$$

The matrix $A - B^T C^{-1} B$ is the voltage-to-current map. In our example

$$A - B^T C^{-1} B = (8/5) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

In fact, for any circuit the voltage to current map is given by

$$A - B^T C^{-1} B = \gamma \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Notice that this form of the matrix implies that $I_1 = I_2$ (the current flowing out the battery at one end flows in on the other side) and that if $b_1 = b_2$ then the currents are zero. The number

$$R = \frac{1}{\gamma}$$

is the ratio of the applied voltage to the resulting current, is the effective resistance of the network between the two nodes.

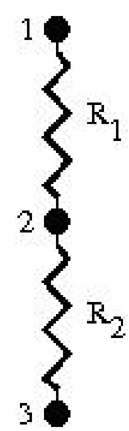
So in our example circuit, the effective resistance between nodes 1 and 2 is 5/8.

If the battery voltages are $b_1 = 0$ and $b_2 = b$ then the voltages at the remaining nodes are

$$\begin{bmatrix} v_3 \\ v_4 \end{bmatrix} = -C^{-1} B \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} b$$

II.2.11 Two resistors in series

Let's do a trivial example where we know the answer. If we connect two resistors in series, the resistances add, and the effective resistance is $R_1 + R_2$. The graph for this example looks like



The Laplacian for this circuit is

$$L = \begin{bmatrix} \gamma_1 & -\gamma_1 & 0 \\ -\gamma_1 & \gamma_1 + \gamma_2 & -\gamma_2 \\ 0 & -\gamma_2 & \gamma_2 \end{bmatrix}$$

with $\gamma_i = 1/R_i$, as always. We want the effective resistance between nodes 1 and 3. Although it is not strictly necessary, it is easier to see what the submatrices A, B and C are if we reorder the vertices so that the ones we are connecting, namely 1 and 3, come first. This reshuffles the rows and columns of L yielding

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 2 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 2 \end{array} & \begin{bmatrix} \gamma_1 & 0 & -\gamma_1 \\ 0 & \gamma_2 & -\gamma_2 \\ -\gamma_1 & -\gamma_2 & \gamma_1+\gamma_2 \end{bmatrix} \end{array}$$

Here we have labelled the re-ordered rows and columns with the nodes they represent. Now the desired submatrices are

$$A = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} \quad B = \begin{bmatrix} -\gamma_1 & -\gamma_2 \end{bmatrix} \quad C = \begin{bmatrix} \gamma_1 + \gamma_2 \end{bmatrix}$$

and

$$A - B^T C^{-1} B = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} - \frac{1}{\gamma_1+\gamma_2} \begin{bmatrix} \gamma_1^2 & \gamma_1\gamma_2 \\ \gamma_1\gamma_2 & \gamma_2^2 \end{bmatrix} = \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

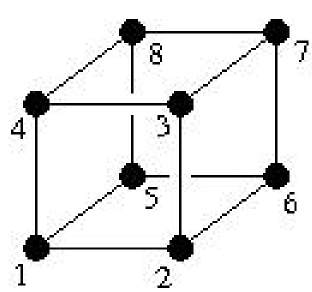
This gives an effective resistance of

$$R = \frac{\gamma_1+\gamma_2}{\gamma_1\gamma_2} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} = R_1 + R_2$$

as expected.

II.2.12 Example: a resistor cube

Hook up resistors along the edges of a cube. If each resistor has resistance $R_i = 1$, what is the effective resistance between opposite corners of the cube?



We will use MATLAB/Octave to solve this problem. To begin we define the Laplace matrix `L`. Since each node has three edges connecting it, and all the resistances are 1, the diagonal entries are all 3. The off-diagonal entries are -1 or 0, depending on whether the corresponding nodes are connected or not.

```
>L=[3 -1 0 -1 -1 0 0 0;-1 3 -1 0 0 -1 0 0;
0 -1 3 -1 0 0 -1 0;-1 0 -1 3 0 0 0 -1;
-1 0 0 0 3 -1 0 -1;0 -1 0 0 -1 3 -1 0;
0 0 -1 0 0 -1 3 -1;0 0 0 -1 -1 0 -1 3];
```

We want to find the effective resistance between 1 and 7. To compute the submatrices A, B and C it is convenient to re-order the nodes so that 1 and 7 come first. In MATLAB/Octave, this can be achieved with the following statement.

```
>L=L([1,7,2:6,8],[1,7,2:6,8]);
```

In this statement the entries in the first bracket `[1,7,`26,8]: indicates the new ordering of the rows. Here 26: stands for `2,3,4,5,6`. The second bracket indicates the re-ordering of the columns, which is the same as for the rows in our case.

Now it is easy to extract the submatrices `A`, `B` and `C` and compute the voltage-to-current map `DN`

```
>N = length(L);
>A = L(1:2,1:2);
>B = L(3:N,1:2);
>C = L(3:N,3:N);
>DN = A - B'*C^(-1)*B;
```

The effective resistance is the reciprocal of the first entry in `DN`. The command `format rat` gives the answer in rational form. (Note: this is just a rational approximation to the floating point answer, not an exact rational arithmetic as in Maple or Mathematica.)

```
>format rat
>R = 1/DN(1,1)

R = 5/6
```

Summary: Math Concepts

- The incidence matrix of a graph.
- Interpretations for each of the four subspaces in terms of voltage and current vectors.
- The dimensions of each subspace.
- The Laplace operator for a graph.
- The nullspace of L.
- Calculating the voltages and currents when a battery is attached.
- Calculating the effective resistance.

Summary: MATLAB/Octave Concepts

- Re-ordering the rows and columns of a matrix
- Extracting submatrices
- Using `format rat`