Ex Last time we solved

$$\frac{dx}{dt} = -3x - 2y$$

$$\frac{dy}{dt} = -2x - 6y$$

The general solution was

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} e^{-7t}.$$

Sketch these solutions in the phase plane x-y.

Ex Find the general solution of

$$\frac{dx}{dt} = -2y$$

$$\frac{dy}{dt} = -2x - 3y$$

and sketch the solutions in the plase plane $x - y$. It is given that

$$\lambda_1 = 1 \qquad \underline{k_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -4 \qquad \underline{k_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

are the eigenvalues & eigenvectors of

$$\begin{bmatrix} 0 & -2 \\ -2 & -3 \end{bmatrix}.$$

Ex For the previous problem, solve the initial value problem

$$x(0) = 1, \quad y(0) = 0.$$

5 Minute Mathematics Break.

Consider $\underline{x}' = \mathbb{A}\underline{x}$ where $\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

Sketch solutions in the phase plane x-y for the following cases:

(i) $\lambda_1 < \lambda_2 < 0$ (both eigenvalues negative, λ_1 is more negative).

(ii) $\lambda_1 < 0 < \lambda_2$

(iii) $0 < \lambda_1 < \lambda_2$ (both positive, λ_2 is more positive).

where $\lambda_{1,2}$ are the (distinct) real eigenvalues of $\mathbb{A}$.

Ex Find the general solution of

$$\frac{dx}{dt} = -x + 2y$$

$$\frac{dy}{dt} = -2x - y$$

and sketch solutions in the phase plane.

Ex Find the general solution of

$$\frac{dx}{dt} = x$$

$$\frac{dy}{dt} = 2x + y - 2z$$

$$\frac{dz}{dt} = 3x + 2y + z.$$

5 Minute Mathematics Break

Consider $\underline{x}' = A\underline{x}$ where $\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Here, A has complex eigenvalues

$$\lambda = a \pm ib.$$

Sketch solutions in the phase plane x-y for the following cases:

(i) $a < 0$

(ii) $a = 0$

(iii) $a > 0$.

You may assume the trajectories move clockwise in time.

5 Minute Mathematics Break

Consider $\ddot{x} + 2\dot{x} + x = 0.$ (1)

The auxilliary equation is

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0 \implies r = -1 \text{ repeated.}$$

So $x(t) = c_1 e^{-t} + c_2 t e^{-t}$ (2)

Write (1) as a first order system.

$$\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = A \begin{bmatrix} x \\ \dot{x} \end{bmatrix}. \qquad (3)$$

$\uparrow$

find A.

Do an eigenanalysis of A. Write the vector form of the general solution of (3) (just $\begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ where x is given in (2)).

If $\mathbb{A}$ has a repeated (twice) eigenvalue λ but only one eigenvector (direction) $\underline{k}$ associated with λ, then

$$\underline{k}\, e^{\lambda t} \qquad \text{and} \qquad (\underline{k}\, t + \underline{v})\, e^{\lambda t}$$

are both solutions of the homogeneous problem

$$\dot{\underline{x}} = \mathbb{A}\, \underline{x}$$

when $\underline{v} \neq \underline{0}$ solves $(\mathbb{A} - \lambda \mathbb{I})\, \underline{v} = \underline{k}$.

$\underline{\text{Ex}}$ Find the general solution to

$$\underline{x}' = \underbrace{\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}}_{\mathbb{A}} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Eigen-analysis of $\mathbb{A}$:

$\lambda_1 = -1$, $\underline{k_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

$\lambda_2 = 3$ $\underline{k_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$\underline{\text{Ex}}$ Find the general solution to

$$\underline{x}' = \underbrace{\begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}}_{\mathbb{A}} \underline{x} + \begin{bmatrix} 0 \\ -e^{-3t} \end{bmatrix}.$$

Eigen analysis of $\mathbb{A}$:

$\lambda_1 = -1 + 2i$ $\underline{k_1} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}.$

$\underline{\text{Ex}}$ Find the general solution to

$$\underline{x}' = \underbrace{\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}}_{\mathbb{A}} \underline{x} + \begin{bmatrix} 2e^{-t} \\ -e^{-t} \end{bmatrix}.$$

Eigenanalysis of $\mathbb{A}$:

$\lambda_1 = 1 \qquad \underline{k_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1 \qquad \underline{k_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$