Mech 221 Week 6 Mathematics Tutorial

A) Consider $x(t)$ satisfying

$$\left.\begin{aligned} x'' + x &= 0 \\ x(0) &= 1 \\ x'(0) &= 0 \end{aligned}\right\} \text{ undamped spring.}$$

Show that $x(t) = \cos t$ solves this problem

B) With units, the same problem is

$$m y'' + k y = 0$$

(m: mass; k: spring constant)

$$y(0) = y_0 \quad \text{initial extension}$$

$$y'(0) = 0$$

a) what are the units of k?

b) scale y & t to arrive at the problem in A).

C) Show that

$$\frac{1}{2}\left\{ (x')^2 + x^2 \right\} \leftarrow \text{scaled Energy}$$

is constant in time for all solutions of $x'' + x = 0$.

D) Rewrite $x'' + x = 0$ as a first order system for x and x'. Write the system as

$$\underbrace{\begin{pmatrix} x \\ x' \end{pmatrix}'}_{\underline{y}} = A \begin{pmatrix} x \\ x' \end{pmatrix}$$

A: 2×2 matrix, determine A.

$$\underline{y} = \begin{bmatrix} x \\ x' \end{bmatrix}$$

E) Apply Forward Euler to the problem in D). Show that the method can be written

$$\underline{Y}_{j+1} = \mathbb{B}_{FE}\,\underline{Y}_j$$

↑ 2×2 matrix depends on h, determine $\mathbb{B}_{FE}$.

F) Take N steps with $h = 2\pi/N$.

$$\underline{Y}_N \approx \mathbb{B}_{FE}^{N}\,\underline{Y}_0 \leftarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

↑ with $h = 2\pi/N$.

Compare $\underline{Y}_N$ with the exact $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (after one period, same as initial conditions).

$\mathbb{B}_{FE}^{N}$ can be computed in MATLAB.

Note that FE reduces energy and introduces a phase error. Discuss the implications of this if FE is used to compute the problem for <u>many</u> periods, even if h is small.

G) Repeat E & F for Backward Euler time stepping.