Mech 221 Math Suggested Problems, week #9 solutions.

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§4.8 #7

$$y' = \underbrace{\begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}}_{A} y + \begin{bmatrix} \sin t \\ 0 \end{bmatrix} \qquad y(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Do eigen analysis of A

$$\det(A - \lambda I) = \det\begin{bmatrix} -3-\lambda & -2 \\ 4 & 3-\lambda \end{bmatrix} = (-3-\lambda)(3-\lambda) + 8$$

$$= \lambda^2 - 1 \quad \Rightarrow \lambda = \pm 1$$

$\lambda_1 = 1$ $\quad A - \lambda_1 I = \begin{bmatrix} -4 & -2 \\ 4 & 2 \end{bmatrix} \qquad \underline{x}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

$\lambda_2 = -1$ $\quad \begin{bmatrix} -2 & -2 \\ 4 & 4 \end{bmatrix} \qquad \underline{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$$y_c(t) = c_1 e^{t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

trial form $y_p(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \sin t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \cos t = \begin{bmatrix} a_1 \sin t + b_1 \cos t \\ a_2 \sin t + b_2 \cos t \end{bmatrix}$

$$y_p' = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cos t - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \sin t = \begin{bmatrix} a_1 \cos t - b_1 \sin t \\ a_2 \cos t - b_2 \sin t \end{bmatrix}.$$

plug into DE

$$\begin{bmatrix} a_1\cos t - b_1 \sin t \\ a_2 \cos t - b_2 \sin t \end{bmatrix} = \begin{bmatrix} -3(a_1 \sin t + b_1 \cos t) - 2(a_2 \sin t + b_2 \cos t) \\ 4(a_1 \sin t + b_1 \cos t) + 3(a_2 \sin t + b_2 \cos t) \end{bmatrix} + \begin{bmatrix} \sin t \\ 6 \end{bmatrix}$$

Equate coefficients,

cost, 1st component : $a_1 = -3b_1 - 2b_2$

sint, " " : $-b_1 = -3a_1 - 2a_2 + 1.$

cost, 2nd " : $a_2 = 4b_1 + 3b_2$

sint, " " : $-b_2 = 4a_1 + 3a_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 0 \\ 3 & 2 & -1 & 0 & 1 \\ 0 & 1 & -4 & -3 & 0 \\ 4 & 3 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 0 \\ 0 & 2 & -10 & -6 & 1 \\ 0 & 1 & -4 & -3 & 0 \\ 0 & 3 & -12 & -7 & 0 \end{array}\right]$$

order a_1, a_2, b_1, b_2.

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 3 & 2 & 0 \\ 0 & 1 & -4 & -3 & 0 \\ 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right]$$

so $b_2 = 0$, $b_1 = -\frac{1}{2}$, $a_2 = -2$, $a_1 = \frac{3}{2}$.

$$y_p(t) = \begin{bmatrix} \frac{3}{2}\sin t - \frac{1}{2}\cos t \\ -2\sin t \end{bmatrix}.$$

$$y(t) = \begin{bmatrix} -C_1 e^{t} - C_2 e^{-t} + \frac{3}{2}\sin t - \frac{1}{2}\cos t \\ 2C_1 e^{t} + C_2 e^{-t} - 2\sin t \end{bmatrix}$$

match initial conditions:

$$\left.\begin{aligned} -C_1 - C_2 - \tfrac{1}{2} &= 0 \\ 2C_1 + C_2 &= 0. \end{aligned}\right\} \quad \begin{aligned} C_1 &= \tfrac{1}{2} \\ C_2 &= -1. \end{aligned}$$

so $$y(t) = \begin{bmatrix} -\frac{1}{2} e^{t} + e^{-t} + \frac{3}{2}\sin t - \frac{1}{2}\cos t \\ e^{t} - e^{-t} - 2\sin t \end{bmatrix}.$$

#19. $$\underline{y}' = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\mathbb{A}} \underline{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Eigenanalysis of $\mathbb{A}$:

$$\det\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1, \qquad \lambda = \pm i.$$

$$\underset{\substack{\uparrow \\ \lambda_1}}{\mathbb{A} - i\,\mathbb{I}} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \qquad \underline{x_1} = \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

$$\underline{y_1}(t) = e^{it}\begin{bmatrix} -i \\ 1 \end{bmatrix} = (\cos t + i\sin t)\begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Real part $\begin{bmatrix} +\sin t \\ \cos t \end{bmatrix}$, Imaginary part $\begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$.

solution matrix

$$\Psi(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

[Convenient order & multiple of columns since Ψ is now a rotation matrix].

$$\Psi^{-1}(s) = \begin{bmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{bmatrix}$$

using properties of rotation matrices.

<u>Note</u>: $\Psi^{-1}(0) = \mathbb{I}$

$$y(t) = \Psi(t) \underset{\begin{bmatrix} 0 \\ 1 \end{bmatrix}.}{y_0} + \Psi(t) \underbrace{\int_0^t \Psi^{-1}(s)\, g(s)\, ds}_{I.} \quad ★.$$

$$I = \int_0^t \begin{bmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} ds$$

$$= \int_0^t \begin{bmatrix} 2\cos s - \sin s \\ 2\sin s + \cos s \end{bmatrix} ds = \begin{bmatrix} 2\sin s + \cos s \\ -2\cos s + \sin s \end{bmatrix} \Bigg|_0^t$$

$$= \begin{bmatrix} 2\sin t + \cos t - 1 \\ -2\cos t + \sin t + 2 \end{bmatrix}.$$

Now back to ★,

$$y(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} 2\sin t + \cos t - 1 \\ -2\cos t + \sin t + 2 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} \sin t + 2\sin t\cos t + \cos^2 t - \cos t - 2\sin t\cos t + \sin^2 t + 2\sin t \\ \cos t - 2\sin^2 t - \sin t\cos t + \sin t - 2\cos^2 t + \sin t\cos t + 2\cos t \end{bmatrix}.$$

$$= \begin{bmatrix} 1 - \cos t + 3\sin t \\ -2 + 3\cos t + \sin t \end{bmatrix}.$$