Mech 221 Math week #8
Suggested problems Solutions

#1. $A = \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & t^2 & 1 \end{bmatrix}$.

$\det A = 1$ for all t, given functions are linearly independent (l.i.).

[Here, I used some facts from Math 152: if A is a square matrix and $\det A \neq 0$ then the columns are l.i.].

§4.3, #31 $\psi = \begin{bmatrix} e^t & e^{-2t} \\ 0 & -3e^{-2t} \end{bmatrix}$. $A = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$.

$\psi' = \begin{bmatrix} e^t & -2e^{-2t} \\ 0 & 6e^{-2t} \end{bmatrix}$. $A\psi = \begin{bmatrix} e^t & -2e^{-2t} \\ 0 & 6e^{-2t} \end{bmatrix}$.

$\psi' = A\psi$ ✓ ψ is a fundamental matrix.

$$\begin{bmatrix} 2e^{-2t} & 0 \\ -6e^{-2t} & 0 \end{bmatrix} = \begin{bmatrix} e^t & e^{-2t} \\ 0 & -3e^{-2t} \end{bmatrix} \underbrace{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}}_{C}$$

$c_{11} = 0$, $c_{21} = 2$

$c_{12} = c_{22} = 0$.

So $\mathbb{C} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$ $\det \mathbb{C} = 0$,

so $\tilde{\Psi}$ is not a fundamental soln.

[Could also note that $\det \tilde{\Psi} = 0$ for all t].

#3. $\mathbb{A} = \begin{bmatrix} 1 & 2 \\ -4 & 7 \end{bmatrix}$.

$$\det(\mathbb{A} - \lambda \mathbb{I}) = \begin{vmatrix} 1-\lambda & 2 \\ -4 & 7-\lambda \end{vmatrix} = (1-\lambda)(7-\lambda) + 8$$
$$= \lambda^2 - 8\lambda + 15$$
$$= (\lambda - 3)(\lambda - 5).$$

$\lambda_1 = 3$ $\quad A - \lambda \mathbb{I} = \begin{bmatrix} -2 & 2 \\ -4 & 4 \end{bmatrix}$ $\quad$ e vector $\underline{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\lambda_2 = 5$ $\quad \begin{bmatrix} -4 & 2 \\ -4 & 2 \end{bmatrix}$ $\quad \underline{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$$\underline{y}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t}.$$

$\underline{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ solves $\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 1 & 2 & 11 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 4 \end{array}\right]$.

$c_2 = 4$, $c_1 = 3$

so $\underline{y}(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t}$

#4 $A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 2 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(3-\lambda)$$

$\lambda = 1, 2, 3$

$\lambda_1 = 1$ $\quad A - \lambda I = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \underline{X_1} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$

$\lambda_2 = 2$ $\quad \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad \underline{X_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$

$\lambda_3 = 3$ $\quad \begin{bmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad \underline{X_3} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$

$$\underline{y}(t) = C_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + C_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{3t}$$

C_1, C_2, C_3 solve $\left[\begin{array}{ccc|c} -1 & 1 & 1 & 4 \\ -1 & 0 & 1 & 3 \\ 1 & 0 & 0 & -1 \end{array}\right] \sim \cdots$

$$\sim \cdots \left[\begin{array}{ccc|c} 1 & -1 & -1 & -4 \\ 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

$C_3 = 2, \quad C_2 = 1, \quad C_1 = -1.$

$$\underline{y}(t) = -\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{t} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{3t}$$

§4.6 #1. $A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ $\quad \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = \cdots$

$= (2-\lambda)^2 + 1 = 0 \quad \Rightarrow \quad (2-\lambda)^2 = -1$

$2 - \lambda = \pm i$

$\lambda = 2 \pm i.$

$\lambda_1 = 2 + i \quad A - \lambda_1 I = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \quad \underline{x_1} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\lambda_2 = 2 - i \quad \underline{x_2} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ (conjugate).

§4.6 #16 $\lambda_1 = 1 + 5i \quad \underline{x_1}^T = (i, 1, 0, 0).$

$\lambda_2 = 1 + 2i \quad \underline{x_2}^T = (0, 0, i, 1).$

$$\underline{y}(t) = c_1 e^t \begin{bmatrix} -\sin 5t \\ \cos 5t \\ 0 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos 5t \\ \sin 5t \\ 0 \\ 0 \end{bmatrix}$$

$$+ c_3 e^t \begin{bmatrix} 0 \\ 0 \\ -\sin 2t \\ \cos 2t \end{bmatrix} + c_4 e^t \begin{bmatrix} 0 \\ 0 \\ \cos 2t \\ \sin 2t \end{bmatrix}.$$

\# 7. $\mathbb{A} = \begin{bmatrix} -2 & -2 & -9 \\ -1 & 1 & -3 \\ 1 & 1 & 4 \end{bmatrix}$

$$\det(\mathbb{A} - \lambda \mathbb{I}) = \begin{vmatrix} -2-\lambda & -2 & -9 \\ -1 & 1-\lambda & -3 \\ 1 & 1 & 4-\lambda \end{vmatrix}$$

$$= (-2-\lambda)\left[(1-\lambda)(4-\lambda)+3\right] + 2\left[-(4-\lambda)+3\right] - 9\left[-1-(1-\lambda)\right].$$

$$= -\lambda^3 + 3\lambda^2 - 4\lambda + 2 = 0$$

<u>Note</u>: $\lambda_1 = 1$ is a root by inspection

$$\begin{array}{r|l} & -\lambda^2 + 2\lambda - 2 \\ \lambda - 1 & -\lambda^3 + 3\lambda^2 - 4\lambda + 2 \\ & -\lambda^3 + \lambda^2 \\ \hline & 2\lambda^2 \\ & 2\lambda^2 - 2\lambda \\ \hline & -2\lambda \end{array}$$

$\lambda^2 - 2\lambda + 2 = 0$

$\Rightarrow \lambda_{2,3} = 1 \pm i.$

$\lambda_1 = 1$ $\quad A - \lambda_1 I = \begin{bmatrix} -3 & -2 & -9 \\ -1 & 0 & -3 \\ 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\underline{X_1} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$

$\lambda_2 = 1+i$ $\quad \begin{bmatrix} -3-i & -2 & -9 \\ -1 & -i & -3 \\ 1 & 1 & 3-i \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3-i \\ 0 & 1-i & -i \\ 0 & 1+i & +1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 3-i \\ 0 & 1-i & -i \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \underline{X_2} = \begin{bmatrix} \frac{1}{2}(-5+i) \\ \frac{1}{2}(-1+i) \\ 1 \end{bmatrix}.$

or $\underline{X_2} = \begin{bmatrix} (-5+i) \\ (-1+i) \\ 2 \end{bmatrix}$ is a little nicer to work with.

$\lambda_3 = 1-i$ $\quad \underline{X_3} = \begin{bmatrix} -5-i \\ -1-i \\ 2 \end{bmatrix}.$

Now, the general soln is

$$\underline{y}(t) = C_1 e^t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + C_2 e^t \begin{bmatrix} -5\cos t - \sin t \\ -\cos t - \sin t \\ 2\cos t \end{bmatrix} + \cdots$$

$$+ C_3 e^t \begin{bmatrix} -5\sin t + \cos t \\ -\sin t + \cos t \\ 2\sin t \end{bmatrix}.$$

At $t=0$, C_1, C_2, C_3 solve the system

$$\left[\begin{array}{ccc|c} -3 & -5 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 1 & 2 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & 12 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 13 \end{array}\right]$$

So $C_3 = \frac{13}{2}$, $C_2 = \frac{11}{2}$, $C_1 = -9$.

$$y(t) = -9e^t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + \frac{11}{2} e^t \begin{bmatrix} -5\cos t - \sin t \\ -\cos t - \sin t \\ 2\cos t \end{bmatrix}$$

$$+ \frac{13}{2} e^t \begin{bmatrix} -5\sin t + \cos t \\ -\sin t + \cos t \\ 2\sin t \end{bmatrix}.$$