Mech 221 Math Suggested problems solutions
Week #6, Brian Wetton.

§3.1 #3. Solution defined on $(-\infty, -1)$.

§3.2 #5. $y'' - 4y' + 4y = 0$

a) $y_1(t) = e^{2t}$ and $y_2(t) = te^{2t}$ are both solutions (differentiate and plug in to verify).

b) $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t} + 2te^{2t} \end{vmatrix}$$

$$= e^{4t}(1+2t) - 2te^{4t} = e^{4t} \text{ never zero.}$$

c) $y(t) = Ay_1 + By_2$ $\qquad$ $y(0) = 2 \Rightarrow A = 2$

$y'(t) = 2Ae^{2t} + B(1+2t)e^{2t}$

$y'(0) = 0 \Rightarrow \quad 2A + B = 0 \Rightarrow B = -4$

so $y(t) = 2e^{2t} - 4te^{2t}$.

Q #3 Abel's Theorem says $W'(t) = -p(t)W(t)$. Since $p(t) = 0$ in this case, W is constant so $W(4) = -3$.

§3.3 #5 $y'' - y = 0$ $\quad y(0) = 1,\ y'(0) = -1.$

$y = Ae^{-x} + Be^{x}$ $\quad y(0) = 1 \Rightarrow A + B = 1$

$y' = -Ae^{-x} + Be^{x}$ $\quad y'(0) = -1 \Rightarrow -A + B = -1$

$\Rightarrow A = 1,\ B = 0.$

so $y(x) = e^{-x}$ $\quad \lim_{x \to \infty} y(x) = 0$

$\lim_{x \to -\infty} y(x) = \infty.$

#11 $2y'' - 3y' = 0$ $\quad y(-2) = 3,\ y'(-2) = 0.$

$y = A + Be^{3/2\, x}$ $\quad A + Be^{-3} = 3.$

$y' = \frac{3}{2} Be^{3/2\, x}$ $\quad \frac{3}{2} Be^{-3} = 0$

so $B = 0,\ A = 3.$

$\underline{\text{Soln}}$ $y = 3$ for all x $\quad \lim_{x \to \pm\infty} y(x) = 3.$

#12 $y'' - 6y' + 8y = 0$ $\quad y(1) = 2,\ y'(1) = -8.$

$r^2 - 6r + 8 = 0$, $\quad (r-4)(r-2) = 0$ $\quad r = 2, 4.$

$y = Ae^{2x} + Be^{4x}.$ $\quad y(1) = 2 \Rightarrow Ae^2 + Be^4 = 2$

$y' = 2Ae^{2x} + 4Be^{4x}$ $\quad y'(1) = -8 \Rightarrow 2Ae^2 + 4Be^4 = -8$

Solve to get

$A = 8e^{-2}$, $B = -6e^{-4}$, so

$y(x) = 8e^{2(x-1)} - 6e^{4(x-1)}$.

$\lim_{x\to\infty} y(x) = \infty$, $\lim_{x\to -\infty} y(x) = -\infty$.

§3.4 #3. $y'' + 6y' + 9y = 0$ $y(0) = 2$, $y'(0) = -2$.

$r^2 + 6r + 9 = 0$, $r = -3, -3$ repeated root.

$y(x) = Ae^{-3x} + Bxe^{-3x}$

$y' = -3Ae^{-3x} + B(1-3x)e^{-3x}$

$y(0) = 2 \Rightarrow A = 2$

$y'(0) = -2 \Rightarrow -6 + B = -2$, $B = 4$.

$y(x) = 2e^{-3x} + \underbrace{4xe^{-3x}}_{}$ dominant term.

$\lim_{x\to\infty} y(x) = 0$, $\lim_{x\to -\infty} y(x) = -\infty$.

#11 $r^2 - 2\alpha + \alpha^2 = 0$

$(r-\alpha)^2 = 0$. $r = \alpha$ repeated root,

$y(x) = Ae^{\alpha x} + Bxe^{\alpha x}$

From the graph, $y(0)=0$, so $A=0$,

$$y(x) = Bxe^{\alpha x}$$

$$y'(x) = B(1+\alpha x)e^{\alpha x}$$ maximum at $x=2$, so $\alpha = -\frac{1}{2}$.

maximum value $8/e$

$$y(2) = B(2)e^{-\frac{1}{2}(2)} = 2Be^{-1}$$

so $B=4$. Now $y'(0)=4$

§ 3.5 #5 $9y''+y=0$ $y(\pi/2)=4$, $y'(\pi/2)=0$

$$y(t) = A\cos(t/3) + B\sin(t/3).$$

$$y(\pi/2) = A\sqrt{3}/2 + B/2 = 4$$

$\cos(\pi/6) = \cos(30°) = \sqrt{3}/2$.

$$y'(t) = -\frac{A}{3}\sin(t/3) + \frac{B}{3}\cos(t/3)$$

$$y'(\pi/2) = -\frac{1}{6}A + \frac{\sqrt{3}}{6}B = 0.$$

$A = 2\sqrt{3}$

$B = 2$.

so $y = 2\sqrt{3}\cos\left(\frac{t}{3}\right) + 2\sin\left(\frac{t}{3}\right)$.

#9. $9y'' + 6y' + 2y = 0$ $\qquad y(3\pi) = 0$

$y'(3\pi) = 1/3$.

$9r^2 + 6r + 2 = 0$.

$$r = \frac{-6 \pm \sqrt{36 - 4(2)(9)}}{18} = -\frac{1}{3} \pm \frac{i}{3}.$$

so $y(t) = e^{-t/3}\left(A\cos(t/3) + B\sin(t/3)\right)$.

$y'(t) = e^{-t/3}\left(\frac{1}{3}\right)\left((B - A)\cos(t/3) - (B+A)\sin(t/3)\right)$

$y(3\pi) = 0 \Rightarrow e^{-\pi}(-1)A = 0 \Rightarrow A = 0$.

$y'(3\pi) = 1/3 \Rightarrow e^{-\pi}\left(\frac{1}{3}\right)(-1)B = 1/3 \Rightarrow B = -e^{\pi}$

so $y = -e^{-t/3 + \pi}\sin(t/3)$.

#25 $y(t) = e^{-t}(\underset{\substack{\uparrow \\ A = -1}}{-\cos t} + \underset{\substack{\uparrow \\ B = \sqrt{3}}}{\sqrt{3}\sin t})$.

$R = \sqrt{A^2 + B^2} = 2$.

$\tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}(-\sqrt{3}) = -\pi/3$. $\leftarrow$ this would correspond to $A > 0$, $B < 0$, so . . .

$\theta = -\pi/3 + \pi = 2\pi/3$.

<u>Note</u>: In matlab, you could take $\theta = \mathrm{atan2}(A, B)$.

so $y = 2e^{-t}(\cos(t - 2\pi/3))$.

§3.5 #29 $R = \frac{1}{2}$.

Distance from zero to adjacent extrema ($\frac{1}{4}$ of wavelength) is $\frac{5\pi}{12} - \frac{\pi}{6} = \frac{\pi}{4}$.

wavelength π, $\beta = 2$. δ is $\frac{5\pi}{6}$ ($\cos(2(t-\frac{5\pi}{12}))$ to shift maximum to origin), so

$y(t) = \frac{1}{2}\cos(2t - 5\pi/6)$.

$y(0) = \frac{1}{2}\cos(-5\pi/6) = \frac{1}{2}\cos(5\pi/6) = -\sqrt{3}/4$.

$y'(t) = -\sin(2t - 5\pi/6)$

$y'(0) = \sin(5\pi/6) = \frac{1}{2}$.

§3.7 #3. $y'' - y' - 2y = 20e^{4t}$ $\quad y(0) = 0$

$y'(0) = 1$.

$y_p(t) = 2e^{4t}$ can be verified by direct substitution.

$r^2 - r - 2 = 0 \Rightarrow r = -1, 2$

$(r-2)(r+1) = 0$.

$y_c(t) = Ae^{-t} + Be^{2t}$.

$y(t) = Ae^{-t} + Be^{2t} + 2e^{4t}$. $\quad A = 1,\ B = -3$.

$$\left.\begin{array}{lll} y(0) = 0 & \Rightarrow & A + B + 2 = 0 \\ y'(t) = 0 & \Rightarrow & -A + 2B + 8 = 1 \end{array}\right\}$$

So $y(t) = e^{-t} - 3e^{2t} + 2e^{4t}$.

#7 $y'' + y = 2t - 3\cos 2t$ $\quad y(0) = 0,\ y'(0) = 0$.

$y_p(t) = 2t + \cos 2t$ can be verified by direct substitution.

$y_c(t) = A\cos t + B\sin t$

$y(t) = A\cos t + B\sin t + 2t + \cos 2t$.

$y(0) = 0 \quad \Rightarrow \quad A + 1 = 0 \quad \Rightarrow \quad A = -1$.

$y'(0) = 0 \quad \Rightarrow \quad B + 2 = 0 \quad \Rightarrow \quad B = -2$.

So $y(t) = -\cos t - 2\sin t + 2t + \cos 2t$.

§ 3.8 #3. $y'' + y = 8e^{t}$. (★)

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$$\left.\begin{aligned} y_p(t) &= ae^{t} \\ y_p'(t) &= ae^{t} \\ y_p''(t) &= ae^{t} \end{aligned}\right\}$$
in (★), equate coefficients of e^{t}: $\quad 2a = 8$
$a = 4$.

$y(t) = \underbrace{A\cos t + B\sin t}_{\text{complementary}} + 4e^{t}$.

#15 $y'' + 4y' + 5y = 2e^{-2t} + \cos t$. (★)

$r^2 + 4r + 5 = 0$

$r = \dfrac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$

$y_c(t) = e^{-2t}(A\cos t + B\sin t)$

$y_p(t) = a\cos t + b\sin t + ce^{-2t}$.

$y_p'(t) = -a\sin t + b\cos t - 2ce^{-2t}$

$y_p''(t) = -a\cos t - b\sin t + 4ce^{-2t}$.

put in (★).

Equate coefficients of

$\cos t$: $-a + 4b + 5a = 1$.

$\sin t$: $-b - 4a + 5b = 0$ $\Rightarrow$ $a = b$

$a = b = \frac{1}{8}$

e^{-2t}: $4c - 8c + 5c = 2$ $\Rightarrow$ $c = 2$

$y(t) = e^{-2t}(A\cos t + B\sin t) + 2e^{-2t} + \frac{1}{8}\cos t + \frac{1}{8}\sin t$.