Mech 221 Math Solutions, Suggested problems week #4

§2.10 #3 $\quad y' = -ty \qquad y(0) = 1.$

a) both linear & separable, use separable soln technique.

$$\frac{dy}{y} = -t\,dt \Rightarrow \ln|y| = -t^2/2 + C.$$

$\uparrow$ $y(0) = 1 \Rightarrow C = 0.$

so $$y(t) = e^{-t^2/2}$$

b) $$y_{k+1} = y_k + h(-kh\,y_k)$$
$$= (1 - kh^2)\,y_k.$$

c) $$y_1 = y_0 = 1.$$
$$y_2 = (1 - (0.1)^2)\,1 = 0.99.$$
$$y_3 = (1 - 2(0.1)^2)\,0.99 = 0.9702$$

d) Exact
$$\left.\begin{aligned} y(0.1) &\approx 0.9950 \\ y(0.2) &\approx 0.9802 \\ y(0.3) &\approx 0.9560 \end{aligned}\right\}$$
note: not a very good approx, h should be smaller.

#2 $y' = y^2 \quad y(0) = 1$

a) separable $\quad \dfrac{dy}{y^2} = dt \quad \Rightarrow \quad -\dfrac{1}{y} = t + c$

$y(0) = 1 \Rightarrow c = -1.$

$$y(t) = \frac{1}{1-t}$$

solution exists $[-\infty, 1)$.

b) FE $\quad y_{k+1} = y_k + h\, y_k^2$

$h = 0.1,$

k	y_k
0	1.
1	1.1
2	1.2210
3	1.3701
4	1.5578
5	1.8005
6	2.1246
7	2.5760
8	3.2397
9	4.2892
10	6.1289
11	9.8852
12	19.6570.

c) All numerical methods make errors, but in this case it is serious - FE predicts the existance of solutions outside the real domain of existance. Using FE to approximate $y(1.2)$ will always give a value, no matter what h.

§7.2 #7 a) $y' = -\frac{t}{y}$ $y(0) = 3$, $C = 9/2$

separable $\frac{y^2}{2} = -t^2/2 + C$

$y = \sqrt{9 - t^2}$ $y(1) = \sqrt{8}$

b) FE $y_{k+1} = y_k - h\left(\frac{hk}{y_k}\right)$

$= y_k - \frac{h^2 k}{y_k}$.

I wrote a MATLAB in line code fragment to compute with $h = \frac{1}{20}$

$$\left.\begin{aligned} y_{20} &\approx 2.8375 \\ \text{Error} = \sqrt{8} - y_{20} &\approx -9.1 \times 10^{-3}. \end{aligned}\right\} \text{FE}$$

Heun

$y^* = y_k - h\left(\frac{hk}{y_k}\right)$ [FE predictor].

$y_{k+1} = y_k - \frac{h}{2}\left(\frac{hk}{y_k} + \frac{h(k+1)}{y^*}\right)$ [corrector].

$$\left.\begin{aligned} y_{20} &\approx 2.8284 \\ \text{Error} = \sqrt{8} - y_{20} &\approx -6.9 \times 10^{-7} \end{aligned}\right\} \text{Heun.}$$

must record more digits than shown

Modified Euler.

$$y^* = y_k - \frac{h}{2}\left(\frac{hk}{y_k}\right)$$

$$y_{k+1} = y_k - h\left(\frac{h(k+\frac{1}{2})}{y^*}\right).$$

$$\left.\begin{array}{l} y_{20} \approx 2.8284 \\ \text{Error} \approx \sqrt{8} - y_{20} \approx -1.4 \times 10^{-5}. \end{array}\right\} \text{ME.}$$

#9 $y' + 2ty = 0 \qquad y(0) = 2.$

a) separable (linear also).

$c = \ln|2|.$

$$\frac{dy}{y} = -2t\,dt \Rightarrow \ln|y| = -t^2 + C$$

$$y = 2e^{-t^2}$$

$$y(1) = 2/e$$

b) FE $y_{k+1} = y_k - 2h^2k\, y_k = (1 - 2h^2k)\, y_k$

$$\left.\begin{array}{l} y_{20} \approx 0.7488 \\ \text{Error } 2/e - y_{20} \approx -0.013 \end{array}\right\} \text{FE}$$

5.

Heun predictor $y_* = (1-2h^2k)\,y_k$

corrector $y_{k+1} = y_k - h^2\left(k y_k + (k+1)\, y_*\right)$

$$\left.\begin{array}{l} y_{20} \approx 0.7364 \\ \text{Error } \ 2/e - y_{20} \approx -6.0\times 10^{-4} \end{array}\right\} \text{Heun.}$$

Modified Euler $y_* = (1-h^2k)\,y_k$

$$y_{k+1} = y_k - 2h^2\left(k+\tfrac{1}{2}\right) y_*$$

$$\left.\begin{array}{l} y_{20} \approx 0.7354 \\ \text{Error } \ 2/e - y_{20} \approx 3.3\times 10^{-4} \end{array}\right\} \text{ME.}$$