Solutions to Mech 2 Math Suggested Problems week #2

#1. $f(0)=0,\ f(\frac{1}{4})=1,\ f(\frac{1}{2})=3,\ f(\frac{3}{4})=6,\ f(1)=8$

a) $h=\frac{1}{4}$, Trapezoidal rule

$$\int_0^1 f(x)\,dx \approx \frac{1}{8}(0)+\frac{1}{4}(1+3+6)+\frac{1}{8}(8)$$
$$= 3\tfrac{1}{2}$$

b) $$f''(\tfrac{1}{4}) \approx \frac{0-2(1)+3}{(1/4)^2} = 16(1)=16.$$

$$f''(\tfrac{1}{2}) \approx \frac{1-2(3)+6}{(1/4)^2} = 16.$$

$$f''(\tfrac{3}{4}) \approx \frac{3-2(6)+8}{(1/4)^2} = -16.$$

c) From data, we have the (rough) estimate $|f''(x)| \le 16$ for $x \in [0,1]$.

So $\frac{16}{12}(1)(\frac{1}{4})^2 \approx \frac{1}{12}$ is an approximate estimate of the error in (a).

#2 $f(a)$

$$f(a+h) \approx f(a) + hf'(a) + \frac{h^2}{2} f''(a) + \frac{h^3}{6} f'''(a) + \cdots$$

$$f(a+3h) \approx f(a) + 3hf'(a) + \frac{9}{2} h^2 f''(a) + h^3 \frac{9}{2} f'''(a) + \cdots$$

$$a f(a) + b f(a+h) + c f(a+3h) \approx f'(a). \quad \star$$

want

Match terms in $\star$

$f(a)$: $\quad a + b + c = 0.$

$f'(a)$: $\quad hb + 3hc = 1. \qquad -6hc = 1, \quad c = -\frac{1}{6h}$

$f''(a)$: $\quad \frac{1}{2} h^2 b + \frac{9}{2} h^2 c = 0 \quad \Rightarrow \quad b = -9c$

$b = \frac{3}{2h}$

$a = -\frac{3}{2h} + \frac{1}{6h} = \frac{-4}{3h}.$

so $\displaystyle f'(a) \approx \frac{-4/3\, f(a) + 3/2\, f(a+h) - 1/6\, f(a+3h)}{h}.$

Further work shows that the error in this formula is approximately $-\frac{1}{2} f'''(a) h^2$.

#3 (taken from Instructor's solution manual).

$\boxed{1.2}$ 10. $y = -e^{-t} + \sin t$

$y' = e^{-t} + \cos t$

$y' + y = \underbrace{e^{-t} + \cos t - e^{-t}} + \sin t$

so $g(t) = \cos t + \sin t$

$y(0) = -1 = y_0$.

11. $y = t^r$, $y' = rt^{r-1}$, $y'' = r(r-1)t^{r-2}$

Thus $t^2 y'' - 2ty' + 2y = 0$

$r(r-1)t^r - 2rt^r + 2t^r = 0$

$\underbrace{[r(r-1) - 2r + 2]} t^r = 0$

$r^2 - 3r + 2 = (r-2)(r-1) = 0$

$r = 1, 2$.

$\boxed{2.1}$ 1. linear, nonhomogeneous

9. nonlinear.

$\boxed{2.2}$ 2. general solution $y = Ce^{-3t}$

$y(0) = -3$, $y(t) = -3e^{-3t}$

16. $y' + e^{-t} y = 0$

$\int e^{-t}\,dt = -e^{-t}$ (integrating factor).

$(e^{-e^{-t}} y)' = 0$

$y = c\, e^{e^{-t}}$

$\boxed{2.9}$ 18. (American standard units - ugh!)

$mg = 180$ lb.

a) $0 \le t \le 10$ $\quad v' = -g \quad v(0) = 0$ (free fall).

at $t = 10$, $v = -320$

b) $10 \le t \le 14$ $\quad mv' + kv = -mg \quad y(14) = 0$.

$v(10) = -320$.

The algebra is easier if t is reset to $[0, 4]$.

$$v(t) = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right) e^{-\frac{k}{m}t}$$

ft/mile

For $mg = 200$, $\frac{200}{k} = 10 \quad \frac{5280}{3600}$ ← s/hour.

$k = \frac{360}{528}(20)$

$v(4) \approx -13.219$ ft/s

real 14

c) $h = -\int_0^4 v(t)\,dt = \left(\frac{mg}{k} t - \left[v_0 + \frac{mg}{k} \right] \left(-\frac{m}{k} \right) e^{-\frac{k}{m} t} \right) \Big|_0^4$

shifted time as in b)

≈ 179.3 ft.

d) $h_{balloon} = 179.3 + \underbrace{\frac{1}{2} g (10)^2}_{1600} \approx 1779$ ft.