Solutions to Mech 2 Math Suggested Problems week #1

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1. $f(0) = 0, \quad f'(0) = 1, \quad f(1) = 2.$

a) Estimate $f(\frac{1}{3})$ using linear approximation,

$$f(x) \approx L(x) = \underbrace{0}_{f(a)} + \underbrace{x}_{f'(a)(x-a)} = x \quad \text{with } a=0.$$

so $f(\frac{1}{3}) \approx L(\frac{1}{3}) = \frac{1}{3}$

b) Estimate $f(\frac{1}{3})$ using linear interpolation

$$f(\tfrac{1}{3}) \approx \frac{1-\frac{1}{3}}{1-0}\underbrace{(0)}_{f(0)} + \frac{\frac{1}{3}-0}{1-0}\underbrace{(2)}_{f(1)} \quad \text{using formula.}$$

$$= \frac{1}{3}(2) = \frac{2}{3}.$$

c) Estimate $f(\frac{1}{3})$ using a quadratic function.

$$Q(x) = C_0 + C_1 x + C_2 x^2 \quad ; \quad Q'(x) = C_1 + 2C_2 x.$$

use data to determine c's.

$$Q(0) = 0 \Rightarrow C_0 = 0$$

$$Q'(0) = 1 \Rightarrow C_1 = 1$$

$$Q(1) = 2 \Rightarrow C_1 + C_2 = 2 \Rightarrow C_1 = 1.$$

so $Q(x) = x + x^2$

$$f\left(\tfrac{1}{3}\right) \approx Q\left(\tfrac{1}{3}\right) = \tfrac{1}{3} + \tfrac{1}{9} = \tfrac{4}{27}.$$

2.a) Follow the (big) hint to estimate r:

N	E_N	$E_{2N} - E_N$	$\frac{E_{2N}-E_N}{E_{4N}-E_{2N}}$	$\log_2$
4	1.5250	−0.1589	1.4471	0.533
8	1.3661	−0.1098	1.4372	0.523
16	1.2563	−0.0764		
32	1.1799	—		

with a bit of imagination, you can imagine $r \approx 0.5$.

b) So $E_N \approx I - \dfrac{C}{\sqrt{N}}$

$$E_{2N} \approx I - \frac{C}{\sqrt{2N}} = I - \frac{C}{\sqrt{2}\sqrt{N}}.$$

An appropriate extrapolation is

$$I \approx \frac{1}{1-\frac{1}{\sqrt{2}}} \underbrace{\left(E_{2N} - \frac{1}{\sqrt{2}} E_N \right)}_{\substack{\text{cancels error} \\ \text{estimates } I - \frac{1}{\sqrt{2}} I}}$$

So, the most accurate approximation of I from the data we can obtain is

$$I \approx \frac{1}{1-\frac{1}{\sqrt{2}}}\left(E_{32} - \frac{1}{\sqrt{2}} E_{16}\right)$$

$$\approx 0.9955$$

[Note: this is a real example in which the computation is approximating the exact $I = 1$].

3. $$S(x) = \frac{h-x}{h} f(0) + \frac{x}{h} f(h).$$

$\uparrow$ linear interpolation to the data.

Consider interpolation at a point $x = c$, $S(c) \approx f(c)$
We are interested in the error

$$f(c) - S(c).$$

Consider the specially constructed function $g(x)$, motivated by the error analysis of the tangent line approximation.

$$g(x) = f(x) - S(x) - \frac{x(x-h)}{c(c-h)}[f(c) - S(c)].$$

$\uparrow$ chosen as a quadratic with value zero at 0 and at h and value 1 at $x = c$.

$g(0)=0, \; g(c)=0, \; g(h)=0$

$g'(\xi_1)=0 \quad g'(\xi_2)=0$ Rollé

for ξ_1 in $(0,c)$ for ξ_2 in (c,h).

so $g''(\xi)=0$ Rollé again

for ξ in (ξ_1, ξ_2) So for sure $\xi \in [0,h]$.

Compute $g''(\xi) = f''(\xi) - 2 \dfrac{f(c)-S(c)}{c \cdot (c-h)}$.

$g''(\xi)=0 \Rightarrow$

$$f(c) - S(c) = \frac{f''(\xi)}{2} c\,(c-h).$$

Consider what we can say about the maximum of $|f(c)-S(c)|$ for all c in $[0,h]$:

$$|f(c)-S(c)| \le \max|f(x)-S(x)|$$

$$\le \frac{1}{2} \underbrace{\max|x(x-h)|}_{\text{elementary optimization, } \le h^2/4} \cdot \max|f''|.$$

$$\le \frac{1}{8} K_2 h^2.$$

Above, all maxima are over $[0,h]$.