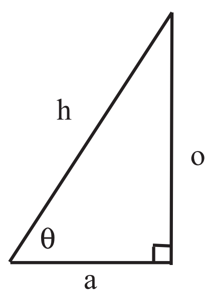
Mech 221 Math Formulae 2008

Trigonometry



$$\begin{aligned}
\sin\theta &= o/h \\
\cos\theta &= a/h \\
\tan\theta &= o/a \\
\tan x &= \frac{\sin x}{\cos x} \\
\sin^2 x + \cos^2 x &= 1 \\
\sin(-x) &= -\sin x \\
\cos(-x) &= \cos x \\
\sin(x+y) &= \sin x \cos y + \cos x \sin y \\
\cos(x+y) &= \cos x \cos y - \sin x \sin y \\
e^{ix} &= \cos x + i \sin x
\end{aligned}$$

Quadratic Equation

$$ax^2 + bx + c = 0$$

has roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Newton's Method

If an initial guess x_0 is close enough to a root of a function g, then the iteration formula

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

gives increasingly good estimates x_n of the root.

Elementary Derivatives

$\frac{d}{dx}x^r = rx^{r-1}$ $(r \neq 0)$

$\frac{d}{dx}\sin x = \cos x$

$\frac{d}{dx}\cos x = -\sin x$

$\frac{d}{dx}\tan x = \sec^2 x = 1/\cos^2 x$

$\frac{d}{dx}e^x = e^x$

$\frac{d}{dx}\ln|x| = 1/x$

$\frac{d}{dx}\sin^{-1} x = 1/\sqrt{1-x^2}$

$\frac{d}{dx}\cos^{-1} x = -1/\sqrt{1-x^2}$

$\frac{d}{dx}\tan^{-1} x = 1/(1+x^2)$

Taylor Polynomials and Series

Taylor polynomial approximation:

$$f(x) \approx P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Residual formula

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

where ξ is a point between a and x (that is not known).

Basic Taylor (McLaurin) series:

$$\begin{aligned}
\sin x &= x - \frac{x^3}{6} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{4!} + \cdots \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots \quad \text{for } |x| < 1
\end{aligned}$$

Numerical Integration

Approximations to

$$I = \int_a^b f(x)dx$$

starting from a division of $[a, b]$ into N sub-intervals of equal length $h = (b-a)/N$:

Trapezoidal Rule:

$$I \approx T_N = \frac{h}{2}f(a) + hf(a+h) + hf(a+2h) + \cdots + hf(b-h) + \frac{h}{2}f(b)$$

with error expression

$$I - T_N = -\frac{f''(\xi)}{12}(b-a)h^2$$

Simpson's Rule: (N must be even)

$$\begin{aligned} I \approx S_N &= \frac{h}{3}f(a) + \frac{4h}{3}f(a+h) + \frac{2h}{3}f(a+2h) + \frac{4h}{3}f(a+3h) + \frac{2h}{3}f(a+4h) + \\ &\cdots + \frac{4h}{3}f(b-h) + \frac{h}{3}f(b) \end{aligned}$$

with error expression

$$I - S_N = -\frac{f^{(4)}(\xi)}{180}(b-a)h^4$$

In the error expressions above, ξ is a point between a and b (that is not known).

Numerical Differentiation

Forward: $f'(a) \approx (f(a+h) - f(a))/h$.

Backward: $f'(a) \approx (f(a) - f(a-h))/h$.

Centred: $f'(a) \approx (f(a+h) - f(a-h))/(2h)$.

Centred second derivative: $f''(a) \approx (f(a+h) - 2f(a) + f(a-h))/h^2$.

Approximating Differential Equations

To find approximations y_k to $y(kh)$ where $y(t)$ solves

$$\frac{dy}{dt} = f(y,t)$$

and h is a small fixed step, the following methods can be used:

Forward Euler: $y_{k+1} = y_k + hf(y_k, kh)$

Backward Euler: $y_{k+1} = y_k + hf(y_{k+1}, (k+1)h)$

Linear Interpolation

If $f(a)$ and $f(b)$ are known and c is in $[a,b]$ then

$$f(c) \approx \frac{b-c}{b-a}f(a) + \frac{c-a}{b-a}f(b)$$

Eigen-Analysis

Eigenvalues of a matrix $\mathbf{A}$ are scalar values λ that solve

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0.$$

If λ is an eigenvalue then $\underline{x}$ is called a corresponding eigenvector if $\underline{x}$ is nonzero and solves

$$(\mathbf{A} - \lambda \mathbf{I})\underline{x} = 0$$

(that is, $\mathbf{A}\underline{x} = \lambda \underline{x}$).

Differential Equations

Scalar, linear, first order

$$y' = a(t)y + f(t)$$

with initial data $y(0) = y_o$. Let

$$A(t) = \int_0^t a(\tau)d\tau.$$

The solution is given by

$$y(t) = e^{A(t)}y_o + e^{A(t)} \int_0^t e^{-A(\tau)} f(\tau)d\tau.$$

Scalar, linear, second-order constant coefficient

$$ay'' + by' + cy = f(t)$$

Solution is $y(t) = y_o(t) + y_p(t)$ (homogeneous plus particular).

homogeneous: Solve auxiliary equation:

$$ar^2 + br + c = 0.$$

Three cases:

1. Two distinct real roots r_1 and r_2:

$$y_o(t) = Ae^{r_1 t} + Be^{r_2 t}.$$

2. Repeated real root r:

$$y_o(t) = Ae^{rt} + Bte^{rt}.$$

3. Complex conjugate roots $r = a \pm ib$:

$$y_o(t) = e^{at}(A\cos bt + B\sin bt).$$

particular: If $f(t)$ has one of the following forms, the Method of Undetermined Coefficients can be used:

$f(t)$ is a polynomial in t of order n: take $y_p(t)$ to also be a polynomial in t of order n.

$f(t) = \sin\omega t$ or $f(t) = \cos\omega t$: take

$$y_p(t) = a\sin\omega t + b\cos\omega t.$$

$f(t) = e^{bt}$: take

$$y_p(t) = ae^{bt}.$$

special case (resonance): If any one of the terms in the form for the particular solution above is in the homogeneous solution, multiply the form of $y_p(t)$ above by t until this is no longer true.

solving for the coefficients: Insert the form of $y_p(t)$ into the differential equation to solve for the undetermined coefficients in $y_p(t)$. Then (and only then) find A and B from the complete solution $y = y_o + y_p$ using the initial (or boundary) data.

Vector, first order, linear, homogeneous, constant coefficient (diagonalizable)

$$\underline{y}' = \mathbf{A}\underline{y}$$

with initial data $\underline{y}(0) = \underline{y_o}$. Assume that $\mathbf{A}$ is a diagonalizable matrix,

$$\mathbf{A} = \mathbf{EDE}^{-1}$$

where $\mathbf{D}$ is the diagonal matrix of eigenvalues λ_1, $\lambda_2, \ldots \lambda_n$ and $\mathbf{E}$ is the matrix with the corresponding eigenvectors in columns. Then the solution is

$$\underline{y}(t) = \mathbf{EME}^{-1}\underline{y_o}$$

where $\mathbf{M}$ is the diagonal matrix with entries $e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots e^{\lambda_n t}$.

Special case: Defective Matrix

In the case above when $\mathbf{A}$ is a diagonalizable matrix, the solution $\underline{y}$ is written as a linear combination of terms

$$\underline{x}_i e^{\lambda_i} t$$

where λ_i are the eigenvalues of $\mathbf{A}$ and $\underline{x}_i$ the corresponding eigenvectors. It can happen for some matrices $\mathbf{A}$ (known as defective matrices) that for certain eigenvalues λ that are repeated twice, there is only one eigenvector $\underline{x}$. In this case, find the vector $\underline{z}$ (a generalized eigenvector) that satisfies

$$(\mathbf{A} - \lambda \mathbf{I})\underline{z} = \underline{x}.$$

The solution of

$$\underline{y}' = \mathbf{A}\underline{y}$$

contains linear combinations of

$$\underline{x} e^{\lambda} t \quad \text{and} \quad (t\underline{x} + \underline{z})e^{\lambda t}$$

in this case.