Mech 221 Mathematics Component

Differential Equations

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Lectures 8-9

Outline

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First Order Autonomous Equations

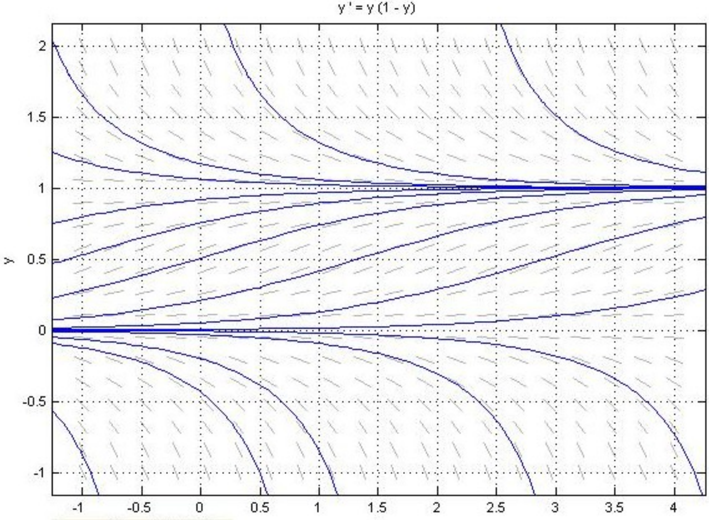
$$y' = f(y) \quad \text{with} \quad y(t_0) = y_0$$

Observe the following:

- The direction field does not depend on t (the definition of autonomous).
- if $f(y) > 0$ then $y' > 0$ and so y is increasing.
- if $f(y) < 0$ then $y' < 0$ and so y is decreasing.
- if $f(y_*) = 0$ for a certain value of y_*, then $y(t) \equiv y_*$ is a solution of the equation (an equilibrium solution).
- Autonomous equations are separable, we can solve them.

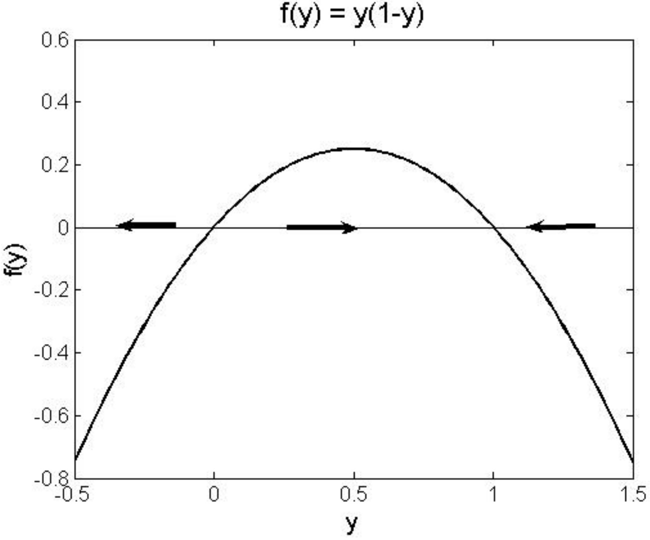
First Order Autonomous Equations-II

An Example: $y' = y(1-y)$ Direction Field



First Order Autonomous Equations-III

An Example: $y' = y(1-y)$ A Different Plot



First Order Autonomous Equations-IV

An Example: $y' = y(1-y)$ Sketch Solutions

First Order Autonomous Equations-V

An Example: $y' = y(1 - y)$ Solution

First Order Autonomous Equations-VI

An Example: $y' = y(1 - y)$ Solution (cont.)

Stability of Equilibrium Points

Stability and Instability

Equilibrium (critical) Points are the roots of $f(y)$. Each equilibrium point is a constant solution of

$$y' = f(y)$$

- An equilibrium solution y_* is called asymptotically stable if all solutions sufficiently close to y_* tend to y_* as $t \to \infty$.
- An equilibrium solution y_* is called unstable if all solutions sufficiently close to y_* move away from y_* as $t \to \infty$.

Stable but not asymptotically stable means that solutions don't move away but also don't get closer (can see periodic solutions around equilibrium points in autonomous systems of equations).

Stability of Equilibrium Points-II

Stability Conditions

If y_* is an equilibrium point ($f(y_*) = 0$) and

- $f(y) < 0$ for $y > y_*$ and
- $f(y) > 0$ for $y < y_*$

then y_* is asympotically stable. This occurs if $f'(y_*) < 0$.

Similarly, if $f'(y_*) > 0$ then the eqilibrium solution y_* is *unstable*.

Stability of Equilibrium Points-III

Example

Find and classify (as stable or unstable) the equilibrium (critical) points of

$$y' = y(1 - y)$$

Stability of Equilibrium Points-IV

Linearization

Let's look more carefully at what happens to solutions near equilibrium solutions. Consider

$$y(t) = y_* + u(t) \text{ where } |u(t)| \text{ is small}$$

in the equation

$$y' = f(y)$$

Stability of Equilibrium Points-V

Linearization (cont).

Another Example

A resistor is connected to a 12V battery. The resistance $R(T)$ in Ohms decreases with temperature T in C as follows:

$$R(T) = \frac{1}{1 + T^2}$$

The resistor has a total heat capacity of 100 J/C. Find and classify equilibrium temperatures for the following 2 cases:

1. The resistor is thermally insulated
2. The resistor is not insulated, is placed in a 0C environment, and it is found that heat loss in W is given by $1000T$.

Another Example-II

Another Example-III

Another Example-IV

Outline

The Logistic Equation

$$y' = ry(K - y)$$

where r and K are given, positive parameters. Questions:

1. What are the equilibrium points?
2. Are they stable or unstable?
3. What are the local decay rates for stable equilibria and local growth rates for unstable equilibria?

The Logistic Equation-II

Questions Answered

The Logistic Equation-III

Questions Answered (cont.)

Scaling

I am not going to solve

$$y' = ry(K - y)$$

because I *claim* I have already solved it since we solved

$$y' = y(1 - y)$$

last lecture (the logistic equation with $r = 1$ and $K = 1$). Let's see why this claim is actually true.

Scaling-II

Scaling-III

Scaling-IV

Scaling-V

Starting with

$$y' = f(y, t)$$

we can scale introduce scaled quantities

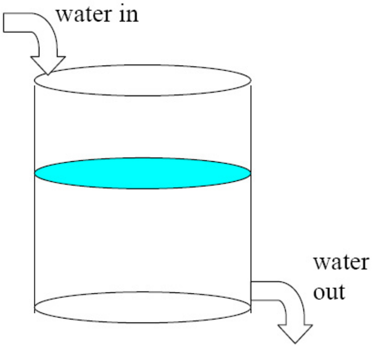
$$\begin{aligned} u &= \frac{y}{Y} \\ s &= \frac{t}{T} \end{aligned}$$

where Y and T are characteristic constant values of y and t. The resulting equation for $u(s)$ can have desireable properties:

- u and s are dimensionless
- Choosing Y and T well can lead to simplifications in the equation
- Universal equations for a class of problems can be identified.
- The relative importance of several terms in an equation can be more easily identified.
- Suitably scaled equations are easier to solve numerically.

Another Example

Suppose a cylindrical water tank of height 1m with cross-sectional area 1 (m^2) is being filled at a rate of 1 litre/s. Water leaks out a small hole in the bottom. When the tank is completely full, the leak rate is 2 litres/s. Assume the draining rate obeys Torricelli's law (rate proportional to square root of water depth).



Another Example-II

Questions

1. Can the tank ever empty?
2. Can the tank ever overflow?
3. Is there a stable equilibrium water depth?
4. What is the equation for $h(t)$, the water depth?
5. What is the function $h(t)$ that describes the water depth if the tank starts half full.

Another Example-III

Answers

Another Example-IV

Answers (cont.)

Another Example-V

General Equation

Derive the general equation in terms of the tank area A, maximum depth D, inflow rate Q_i, outflow rate at maximum depth Q_o. Scale the resulting equation to make it as simple as possible.

Another Example-V

General Equation (cont.)

Another Example-VI

General Equation (cont.)