Mech 221 Mathematics Component

Differential Equations

Brian Wetton

www.math.ubc.ca/∼wetton

Lectures 5-7

Outline

Lecture 5
- Differential Equation Basics
- Linear Constant Coefficient Equations
- Direction Fields

Lecture 6
- Integrating Factor Method
- Proof of Solution Formula
- Examples

Lecture 7
- Equations with Discontinuous Forcing Functions
- Separable Equations
- More Examples

Outline

Lecture 5

- Differential Equation Basics
- Linear Constant Coefficient Equations
- Direction Fields

Lecture 6

- Integrating Factor Method
- Proof of Solution Formula
- Examples

Lecture 7

- Equations with Discontinuous Forcing Functions
- Separable Equations
- More Examples

Differential Equation Basics

Terminology

Differential Equation for unknown function $y(t)$:

$$\frac{dy}{dt} = f(y, t)$$

- This is an Ordinary Differential Equation (ODE). *Ordinary not Partial*. You will study PDEs later.
- This is a first order DE. The order is the maximum number of derivatives in the equation.
- This is a scalar DE. If y were a vector and f a vector function, then it would be a vector DE.
- An extra (initial) condition must be given, usually the value of y at a given value of t.
- The DE and initial condition together make an Initial Value Problem (IVP) that can be solved for $y(t)$.

Differential Equation Basics-II

Linear Equations have the Superposition Property

If the equation has the special form

$$y' + p(t)y = g(t)$$

then it is called a linear equation.

Linear Constant Coefficient Equations

Linear Homogeneous Constant Coefficient Equations

$$y' + ay = 0$$

where a is constant. This is our old friend exponential growth and decay. We can add an initial condition $y(t_0) = y_0$ to make it an IVP.

Solution:

Linear Constant Coefficient Equations-II

Linear Inhomogeneous Constant Coefficient Equations

$$y' + ay = g(t)$$

where $g(t)$ is a given function of t.

- If $y_1(t)$ and $y_2(t)$ are two solutions of the equation, then $y_c(t) = y_1(t) - y_2(t)$ will satisy the homogeneous equation $y' + ay = 0$. We call $y_c(t)$ the *complementary* solution.
- Thus if we find any *particular* solution $y_p(t)$ then the general solution will be of the form

$$y(t) = y_c(t) + y_p(t)$$

- Note that $y_c(t)$ contains a constant, which can be used to fit initial conditions. This is only done after you have the two pieces together in the form above.

Linear Constant Coefficient Equations-III

Method of Undetermined Coefficients

$$y' + ay = g(t)$$

If $g(t)$ has one of the following forms, the Method of Undetermined Coefficients can be used to find the particular solution:

$g(t)$ is a polynomial in t of order n: take $y_p(t)$ to also be a polynomial in t of order n.

$g(t) = \sin \omega t$ or $g(t) = \cos \omega t$: take

$$y_p(t) = a \sin \omega t + b \cos \omega t.$$

$g(t) = e^{bt}$: take

$$y_p(t) = ae^{bt}.$$

Linear Constant Coefficient Equations-IV

Method of Undetermined Coefficients (cont.)

special case (resonance): If any one of the terms in the form for the particular solution above is in the homogeneous solution, multiply the form of $y_p(t)$ above by t until this is no longer true.

solving for the coefficients: Insert the form of $y_p(t)$ into the differential equation and match functions of t to get a linear system for the undetermined coefficients in $y_p(t)$. After the coefficients are determined, then (and only then) find the complete solution $y = y_o + y_p$ using the initial data.

Linear Constant Coefficient Equations-V

MUC Example 1

$$y' + y = t$$

Linear Constant Coefficient Equations-VI

MUC Example 2

$$y' - 5y = e^t$$

with $y(0) = 1$

Direction Fields

Most DEs can't be solved analytically. We will look at numerical methods to find approximate solutions. For scalar, first order equations

$$y' = f(y, t)$$

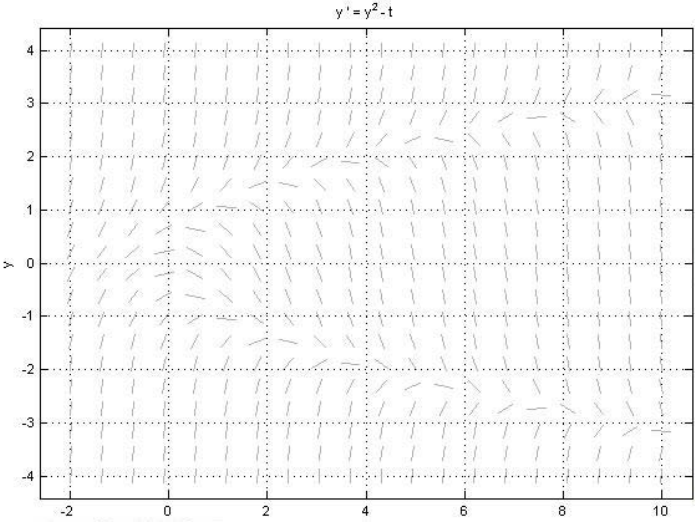
the Direction Field can be plotted. This gives quick qualitative information about the solutions.

1. Draw axes t against y.
2. Make a mesh of points: (t_i, y_j)
3. Draw a small line segment with slope $f(t_i, y_j)$ at each mesh point.

MATLAB software for direction fields can be found at math.rice.edu/∼dfield.

Direction Fields - II

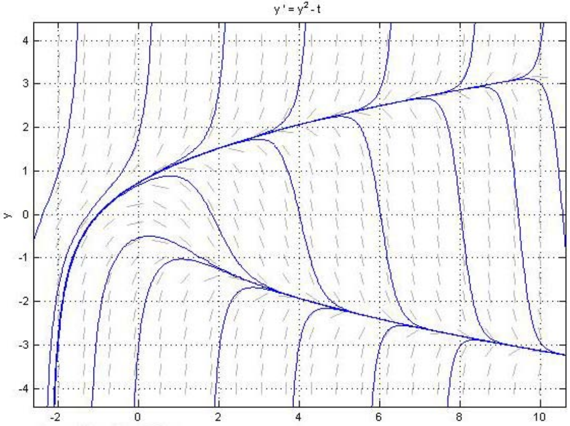
$$y' = y^2 - t$$



Direction Fields - III

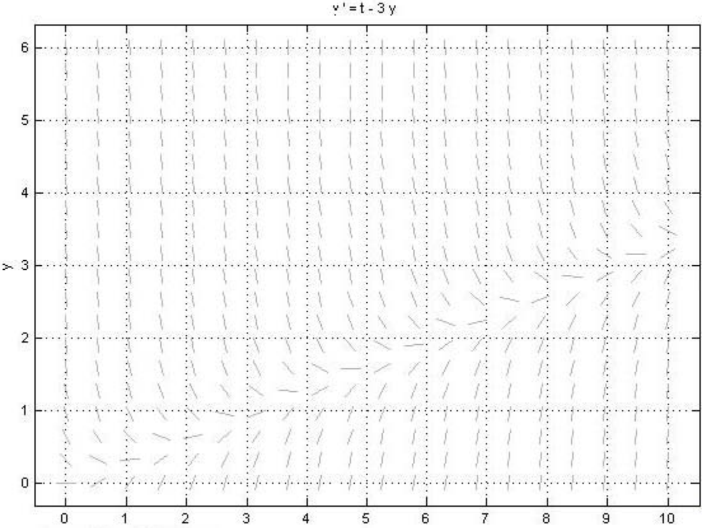
$y' = y^2 - t$ (cont.)

Following the directions in the Direction Field allows you to sketch solution curves.



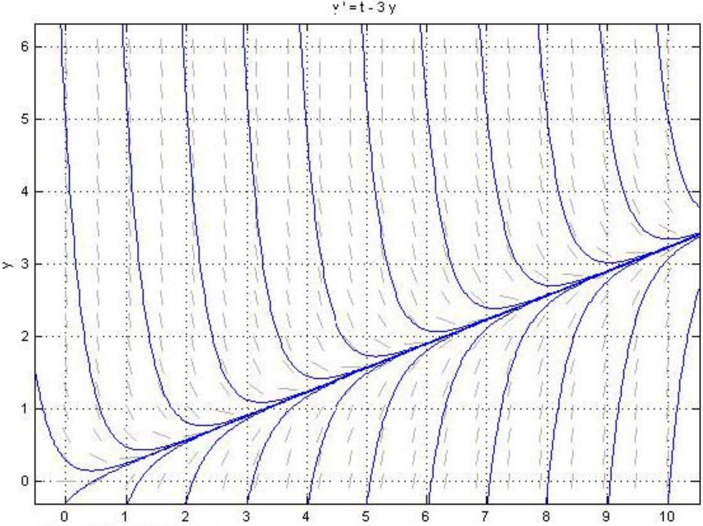
Direction Fields - IV

$$y' = -3y + t$$



Direction Fields - V

$y' = -3y + t$ (cont.)



Direction Fields - VI

Sketching Direction Fields

$$y' = f(t, y)$$

It is possible to sketch a direction field by hand.

1. Sketch the *isocline(s)*, curves where $f(t, y) = 0$.
2. On each side of an isolcline, $y(t)$ will either be increasing or decreasing. Determine which for each region separated by isoclines.
3. Sketch some approximate solution curves.

This technique could also help you match direction fields to equations, a possible homework or test question.

Outline

Lecture 5
Differential Equation Basics
Linear Constant Coefficient Equations
Direction Fields

Lecture 6
Integrating Factor Method
Proof of Solution Formula
Examples

Lecture 7
Equations with Discontinuous Forcing Functions
Separable Equations
More Examples

Integrating Factor Method

General First Order Linear Equation

Last lecture we saw how to solve constant coefficient first order problems with certain specific forms of the RHS. But...

- What if the RHS is not of those specific forms?
- What if the coefficients are not constant?
- What if the equation is non-linear?

General First Order Linear Equation:

$$y' + p(t)y = g(t)$$

Integrating Factor Method-II

Solution Formula

$$y' + p(t)y = g(t)$$

with initial data $y(0) = y_0$. Let

$$P(t) = \int_0^t p(\tau)d\tau.$$

The solution is given by

$$y(t) = e^{-P(t)}y_0 + e^{-P(t)}\int_0^t e^{P(\tau)}g(\tau)d\tau.$$

Note that this solution has the form $y(t) = y_c(t) + y_p(t)$. This is common to all linear DEs.

First, we'll prove this formula, then we'll do examples.

Proof of Solution Formula

Proof of Solution Formula-II

Proof of Solution Formula-III

Examples

Example 1 - $y' - 5y = -e^{5t}$

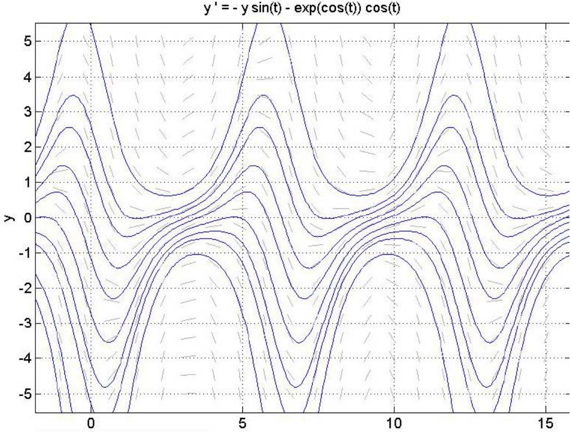
Examples-II

Example 1 - $y' - 5y = -e^{5t}$ using MUC

Examples-III

Example 2 - $y' + y \sin t = \cos t e^{\cos t}$

Looks complicated, let's look at the directions fields first:



Examples-IV

Example 2 - $y' + y\sin t = \cos t e^{\cos t}$ (cont.)

Examples-V

Example 2 - $y' + y\sin t = \cos t e^{\cos t}$ (cont.)

Examples-VI

Review of Solution Technique

1. Put the equation in the correct form

$$y' + p(t)y = g(t)$$

2. Integrate to find

$$\mu(t) = e^{\int^t p(s)ds}$$

3. Multiply the equation by $\mu(t)$ to get

$$\frac{d}{dt}(\mu(t)y(t)) = \mu(t)g(t)$$

4. Integrate to get

$$y(t) = \frac{1}{\mu(t)}\left(\int^t \mu(s)g(s)ds + C\right)$$

5. Fit the constant C using initial data.

Examples-VII

Notes on Solution Technique

- There are two integrations that need to be done (steps 2 and 4). This means there is no guarantee of an analytic solution.
- Convince yourself that the constant of integration in step 2 does not change the solution.
- There is sometimes physical significance to the use of $\mu(t)y(t)$ instead of $y(t)$ to get a simpler equation.
- A unique solution is guaranteed to exist for t values around the data point t_0 up to places (if any) where $p(t)$ or $g(t)$ have singularities.

Outline

Lecture 5
- Differential Equation Basics
- Linear Constant Coefficient Equations
- Direction Fields

Lecture 6
- Integrating Factor Method
- Proof of Solution Formula
- Examples

Lecture 7
- Equations with Discontinuous Forcing Functions
- Separable Equations
- More Examples

Equations with Discontinuous Forcing Functions

Example 1

Solve the IVP:

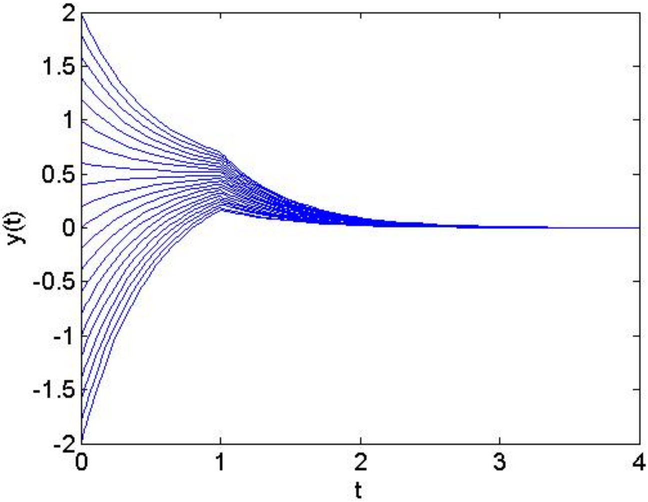
$$y' + 2y = g(t)$$

where $y(0) = a$, given, and

$$g(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & t > 1 \end{cases}$$

Equations with Discontinuous Forcing Functions-II

Example 1 (cont.)



Equations with Discontinuous Forcing Functions-III

Example 1 (cont.)

Equations with Discontinuous Forcing Functions-IV

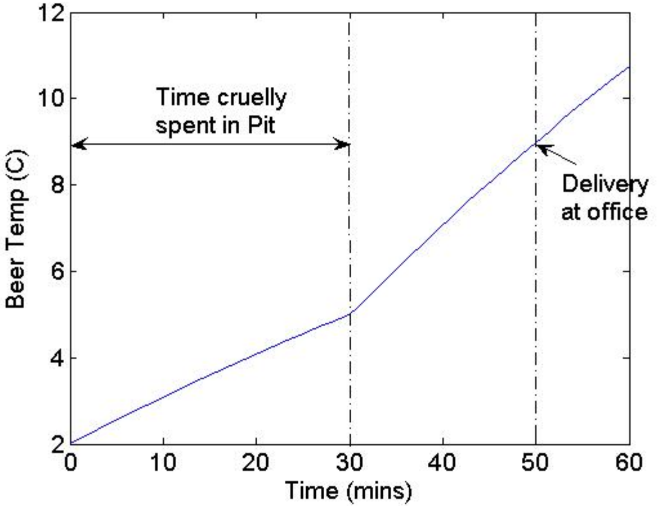
Example 2

It's a hot day and you decide to buy Dr. Frigaard a cold beer. It is poured at 2C in the Pit, which is air-conditioned at 15C. You linger there 30 minutes, and when you leave the beer is 5C. It takes you 20 minutes to walk to Dr. Frigaard's office, outside at 30C. What temperature will the beer be when Dr. Frigaard receives it?

Assume the beer temperature obeys Newton's Law of Cooling.

Equations with Discontinuous Forcing Functions-V

Example 2 (cont.)



Equations with Discontinuous Forcing Functions-VII

Example 2 (cont.)

Equations with Discontinuous Forcing Functions-VIII

Example 2 (cont.)

Separable Equations

Nonlinear Equations

Consider again

$$y' = f(x, y)$$

When $f(x, y)$ is not linear in y we can't solve the equation analytically except in special cases.

separable: Here, f has the special form:

$$f(x, y) = \frac{M(x)}{N(y)}$$

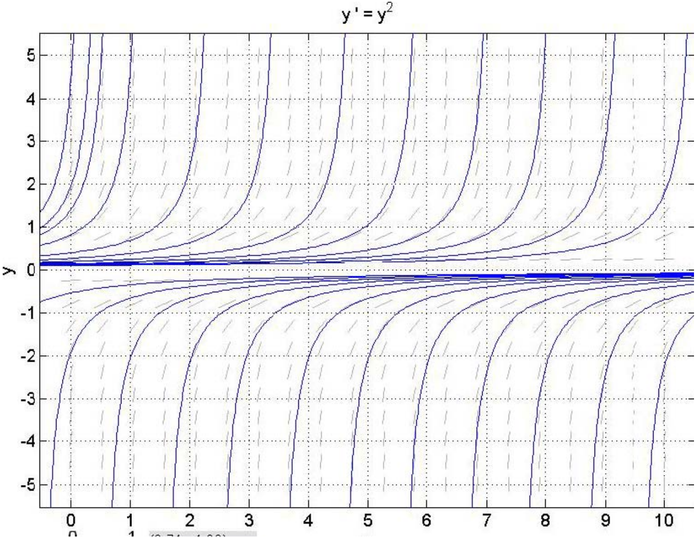
autonomous: A special subset of separable equations where f only depends on y (not x).

Separable Equations-II

Solution Procedure for Separable Equations

Separable Equations-III

Example 1: $y' = y^2$



Separable Equations-IV

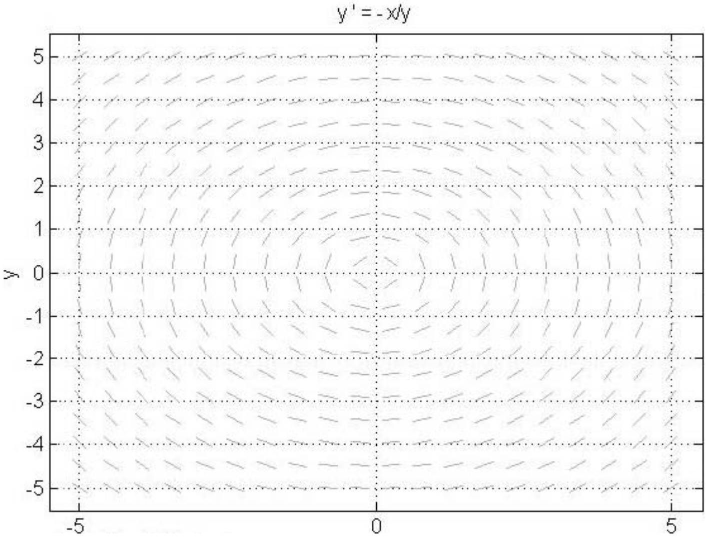
Example 1: $y' = y^2$ (cont.)

Separable Equations-IV

Example 2: $y' = \sqrt{y}$, $y(0) = 0$

Separable Equations-V

Example 3: $y' = -x/y$



Separable Equations-VI

Example 3: $y' = -x/y$

Separable Equations-VII

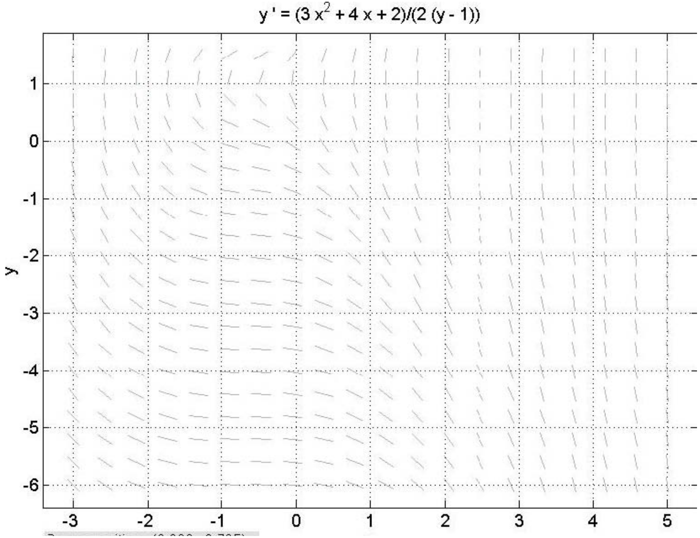
Discussion of Nonlinear Equations

Nonlinear IVP's can have the following behaviour:

- More than one solution.
- Solutions blow up at a certain x or just cease to exist there.

More Examples

Example 4: $y' = (3x^2 + 4x + 2)/(2y - 2)$

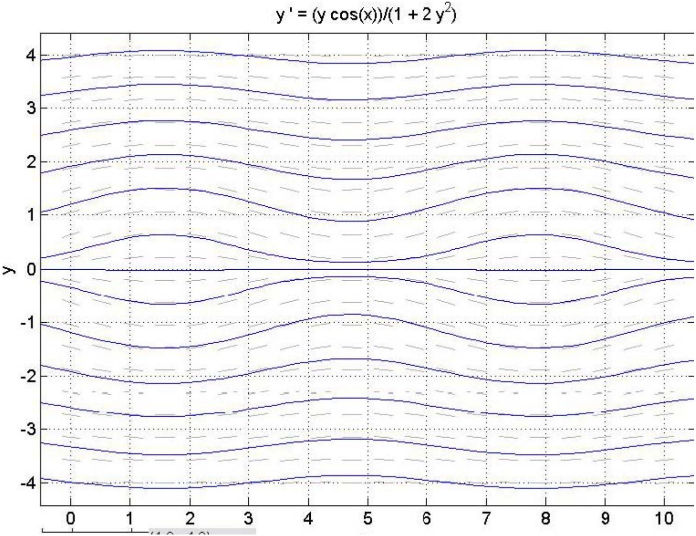


More Examples-II

Example 4: $y' = (3x^2 + 4x + 2)/(2y - 2)$ (cont.)

More Examples-III

Example 5: $y' = (y\cos x)/(1 + 2y^2)$



More Examples-IV

Example 5: $y' = (y \cos x)/(1 + 2y^2)$ (cont.)

More Examples-V

Summary of Solution Procedure

For separable equations:

1. Separate variables
2. Integrate both sides
3. Plug in the IC to determine the unknown constant (or can be done after the next step).
4. Solve the implicit equation for $y(x)$ (if possible).