Autonomous Second Order Differential Equations

These are systems such as:

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned} \tag{1}$$

Frequently these system arise in electrical and mechanical systems, e.g. if x denotes displacement & Newton's 2[nd] law gives a relation of form $x'' = F(x', x)$, then we have the system:

$$\begin{aligned} x' &= y \\ y' &= F(y, x) \end{aligned}$$

Perhaps we want instead to study the forced oscillator

$$x'' = F(x', x) + h(t)$$

but for this understanding of the unforced system is anyway needed

Qualitative understanding:

As with 1[st] order autonomous systems, we can understand much about the behaviour graphically, since (1) does not depend on t. We do this by plotting **phase paths** or solution curves $(x(t), y(t))$ in the (x, y) **phase plane.**

- Note that solution curves are tangent everywhere to the vector

$$(f(x, y), g(x, y))$$

- The slope of the phase path in the phase plane is given by

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)} \tag{2}$$

Sometimes (2) is integrable and we can find phase paths exactly

Phase portraits:

As with 1st order equations, we can plot a type of direction field in the phase plane:

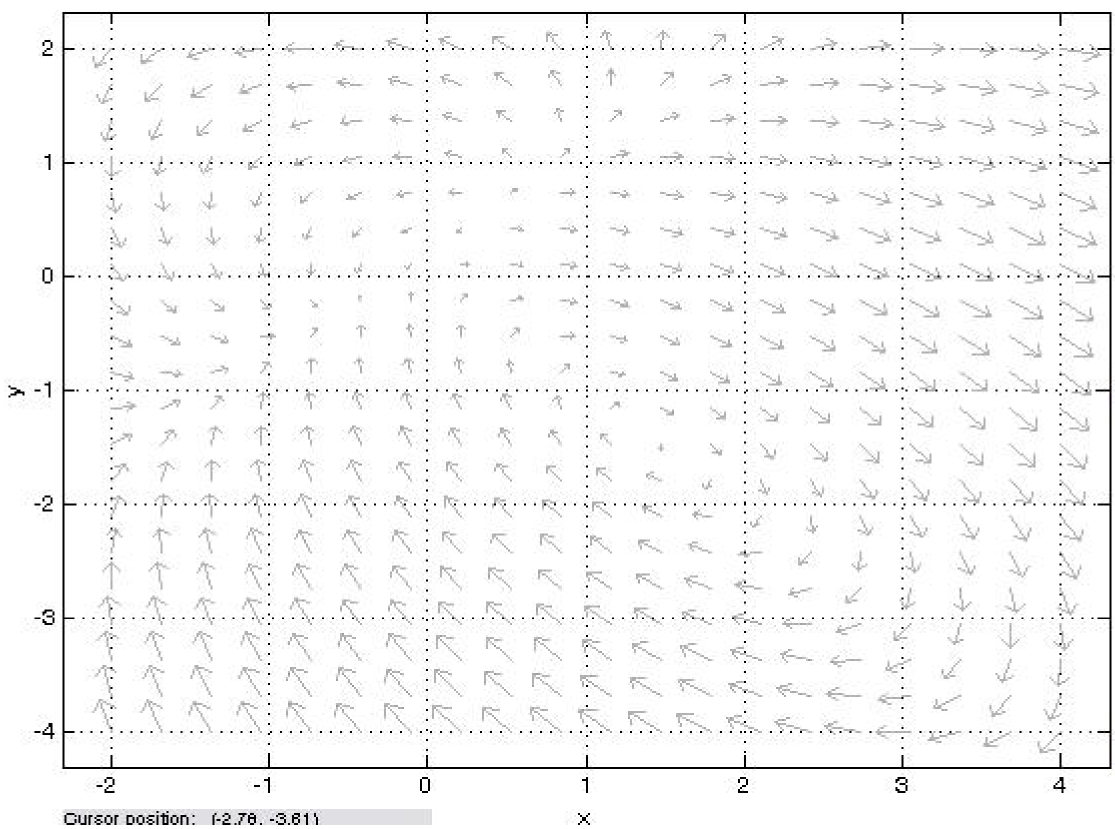
- Set up a grid of points in the phase plane: (x_i, y_j)
- Through each point, draw an arrow with slope given by (2)
- Orient the arrow head in the direction of time, see (1) to determine the sign of (x', y')

There are various software packages that will do this for you, e.g.

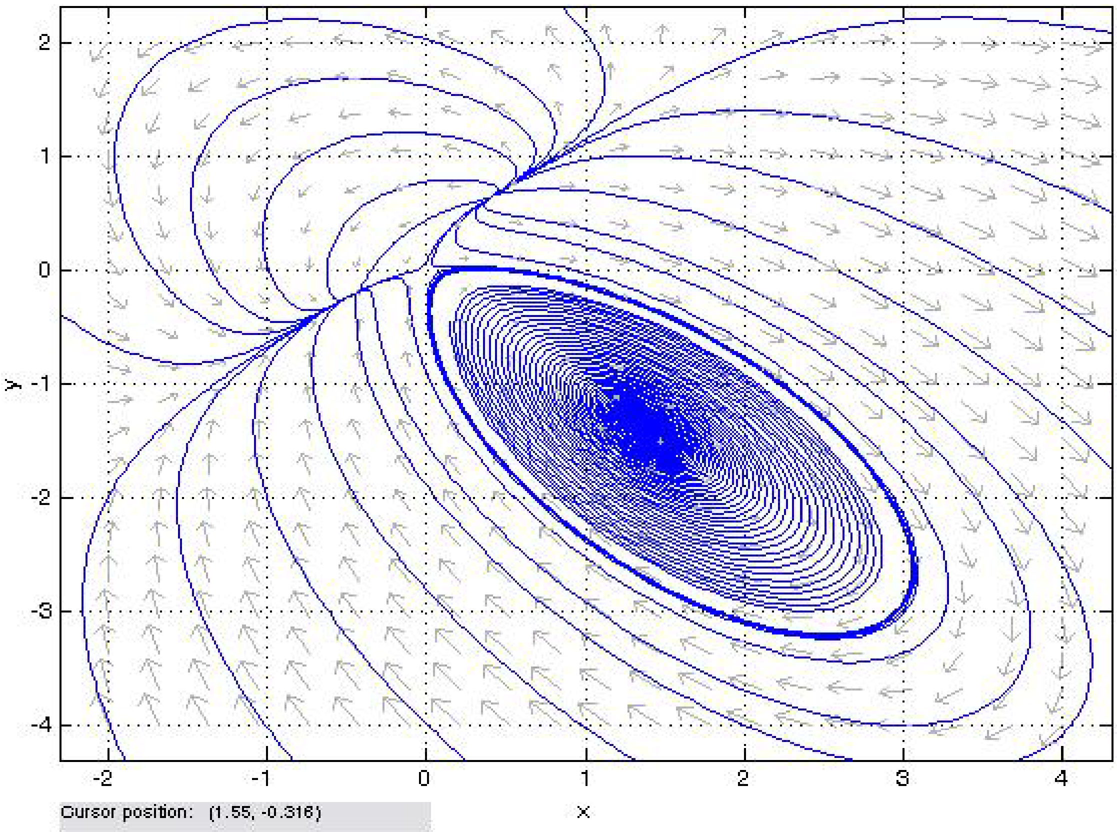
http://math.rice.edu/~dfield/index.html

Example 1: Draw the phase portrait for the system

$$x' = 2x - y + 3\left(x^2 - y^2\right) + 2xy$$
$$y' = x - 3y - 3\left(x^2 - y^2\right) + 3xy$$



The phase portrait also refers to a plot of solution curves:



We can see various long-time behaviours of solutions:

- Periodic paths (stable, asymptotically stable or unstable)
- Critical points
- Paths that go to ∞ at large times
- Can anything else happen?

Critical points (equilibrium points):

- These are stationary solutions of the system (1), i.e. neither $x(t)$ nor $y(t)$ changes with time
- These are found as zeros of the system:

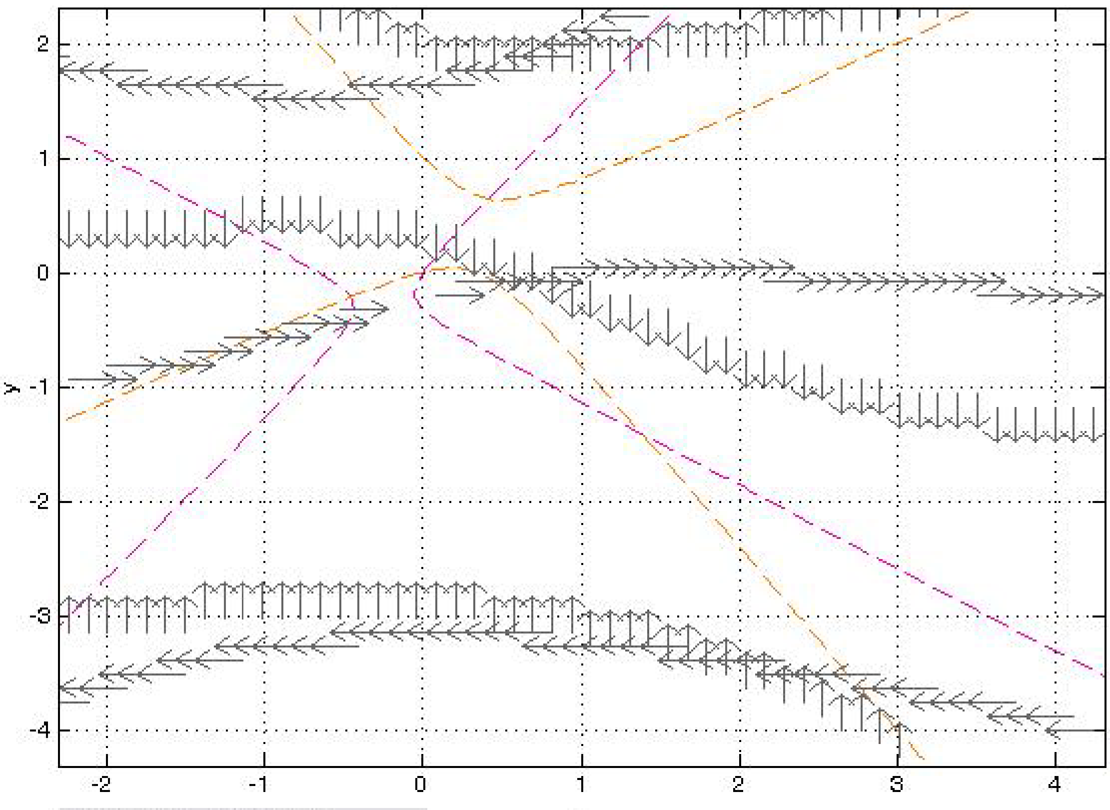
$$\begin{aligned} 0 &= f(x, y) \\ 0 &= g(x, y) \end{aligned} \qquad (3)$$

- In complex systems, we find the critical points graphically as the intersections of the *x* and *y*-**nullclines**

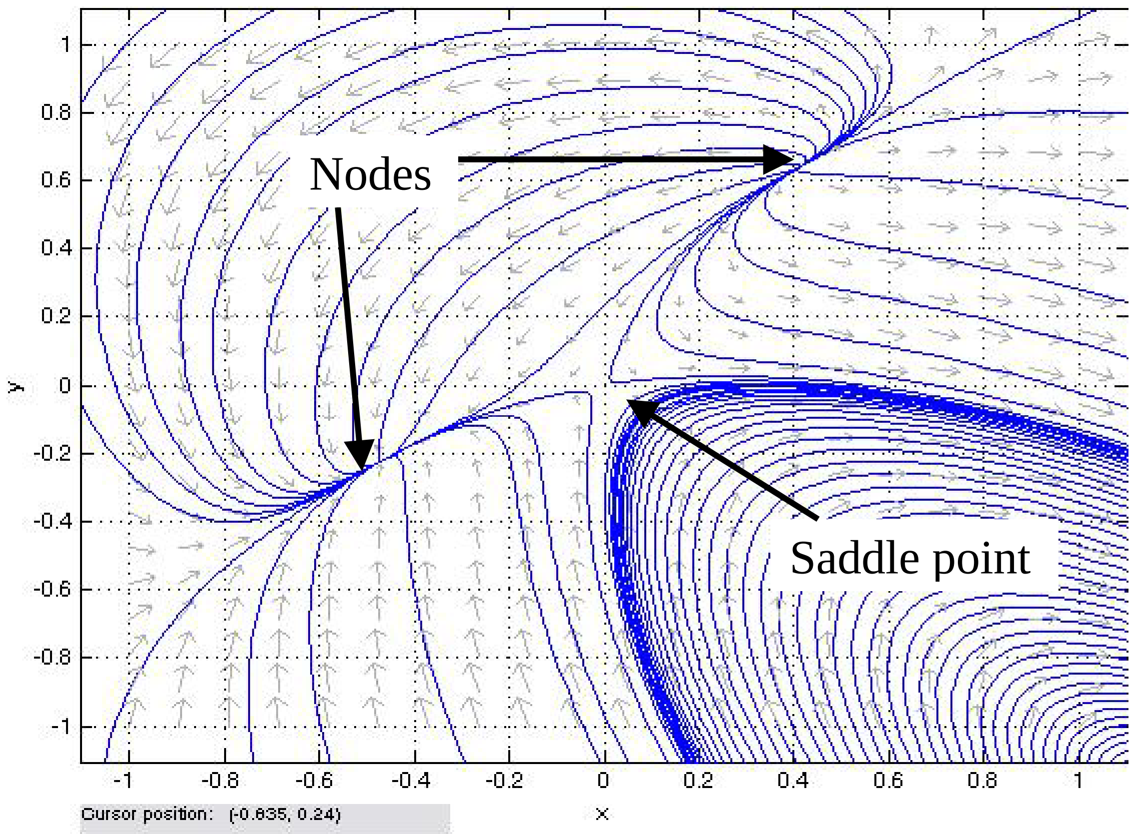
> *x*- **nullcline** is where $0 = f(x, y)$
>
> *y*-**nullclines** is where $0 = g(x, y)$

- Plotting nullclines separates phase plane into regions in which overall direction of phase paths is determined.

$$2x - y + 3(x^2 - y^2) + 2xy = 0$$
$$x - 3y - 3(x^2 - y^2) + 3xy = 0$$



- Appears that there are 4 critical points.
- What is the local behaviour, close to the critical points?
    - Zoom into the square $(x, y) \in [-1,1] \times [-1,1]$



This is reminiscent of phase plane behaviour in 2x2 linear systems?

Linearisation about a critical point:

Suppose we have a critical point $\left(x^*, y^*\right)$, i.e. satisfying (3). Consider what happens when we are close to $\left(x^*, y^*\right)$. Let the solution be:

$$(x(t), y(t)) = \left(x^*, y^*\right) + (u(t), v(t)),$$

where $|(u(t), v(t))| << 1$.

What system does $(u(t), v(t))$ satisfy?

$$\frac{dx}{dt} = \frac{d}{dt}\left(x^* + u\right) = \frac{du}{dt}$$

Perform Taylor expansion in 2 dimensions:

$$f\left(x^* + u, y^* + v\right) = f\left(x^*, y^*\right) + u\frac{\partial f}{\partial x}\left(x^*, y^*\right) + v\frac{\partial f}{\partial y}\left(x^*, y^*\right) + O\left(|(u,v)|^2\right)$$

Retain only the 1st order terms in $|(u(t),v(t))|$:

$$\frac{du}{dt} = u\frac{\partial f}{\partial x}\left(x^*, y^*\right) + v\frac{\partial f}{\partial y}\left(x^*, y^*\right)$$

$$\frac{dv}{dt} = u\frac{\partial g}{\partial x}\left(x^*, y^*\right) + v\frac{\partial g}{\partial y}\left(x^*, y^*\right)$$

This is a constant coefficient 2x2 linear system:

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}\left(x^*, y^*\right) & \frac{\partial f}{\partial y}\left(x^*, y^*\right) \\ \frac{\partial g}{\partial x}\left(x^*, y^*\right) & \frac{\partial g}{\partial y}\left(x^*, y^*\right) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

We know that the character of the linear system close to the critical point $(u(t),v(t)) = (0,0)$ depends only on the eigenvalues of:

$$\begin{pmatrix} \frac{\partial f}{\partial x}\left(x^*, y^*\right) & \frac{\partial f}{\partial y}\left(x^*, y^*\right) \\ \frac{\partial g}{\partial x}\left(x^*, y^*\right) & \frac{\partial g}{\partial y}\left(x^*, y^*\right) \end{pmatrix}$$

Example 2: Show the critical point $\left(x^*, y^*\right) = (0,0)$, of the system

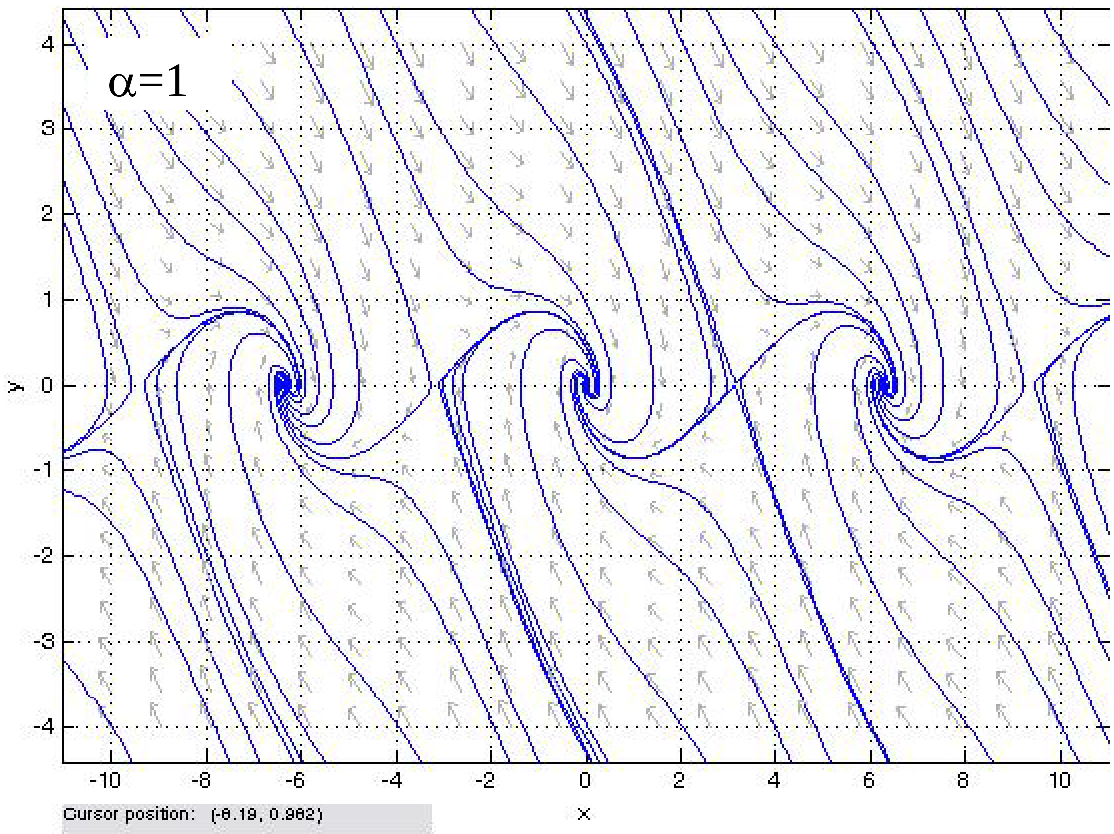
$$x' = 2x - y + 3\left(x^2 - y^2\right) + 2xy$$

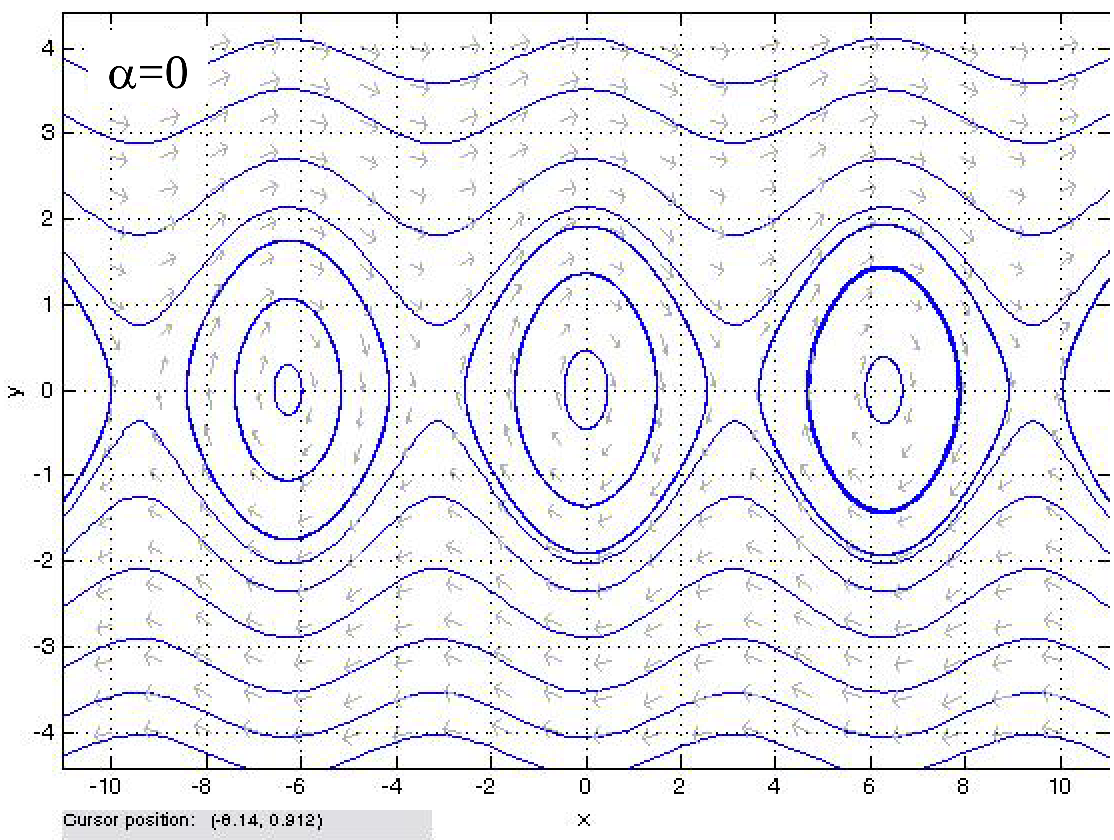
$$y' = x - 3y - 3\left(x^2 - y^2\right) + 3xy$$

is a saddle point.

Example 3: (Damped nonlinear pendulum) For the following system, sketch the nullclines, find and classify the critical points according to a local linear approximation: (i) if $\alpha = 1$; (ii) if $\alpha = 0$.

$$x' = y$$
$$y' = -\sin x - \alpha y$$


α=1
4
3
2
1
y 0
-1
-2
-3
-4
-10
-8
-6
-4
-2
0
2
4
6
8
10
x
Cursor position: (-8.19, 0.962)


α=0
4
3
2
1
y 0
-1
-2
-3
-4
-10
-8
-6
-4
-2
0
2
4
6
8
10
x
Cursor position: (-8.14, 0.912)

Conservative Mechanical Systems

Suppose that x denotes displacement & Newton's 2nd law gives a relation of form $x'' = F(x)$, e.g. a linear spring, then we have the system of form:

$$\begin{aligned} x' &= y \\ y' &= F(x) \end{aligned} \qquad (1)$$

The phase paths have slope

$$\frac{dy}{dx} = \frac{F(x)}{y} \Rightarrow ydy = F(x)dx$$

Integrating both sides of this relation gives:

$$\frac{y^2}{2} + V(x) = E = \text{constant} \qquad (2)$$

where

$$V(x) = -\int^{x} F(s)ds$$

The first term in (2) represents the kinetic energy of the system, $V(x)$ represents potential energy. The total energy E is a constant.

We can see this another way by differentiating (2) with respect to t

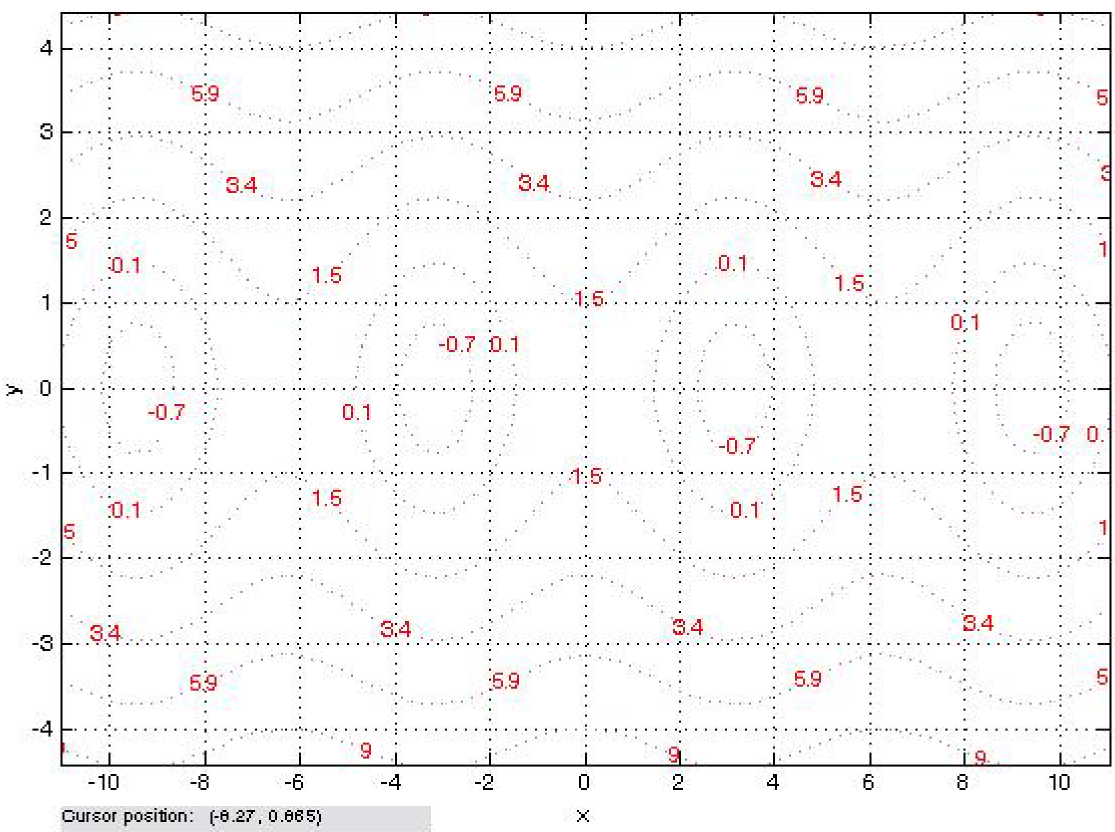
$$\begin{aligned} \frac{dE}{dt} &= yy' + \frac{dV}{dx}x' = yy' - F(x)x' \\ &= yF(x) - F(x)y = 0 \end{aligned}$$

The phase paths of (1) are equivalent to the level lines of (2), which may be an implicit relation.

Example 1: (undamped nonlinear pendulum) Find the total energy for the system:

$$x' = y$$
$$y' = -\sin x$$

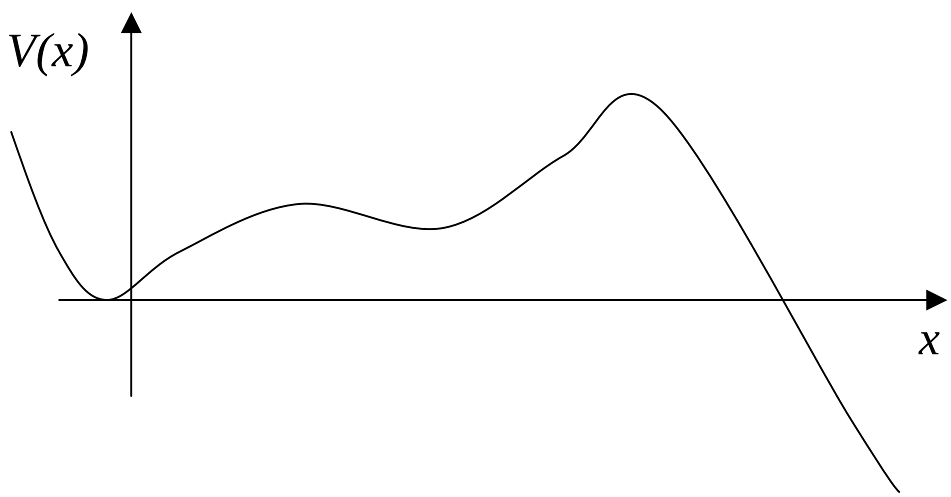
and hence plot the phase paths.



Critical points & the potential energy:

- The critical points of a conservative system such as (1) are where $y = 0$ and where $V(x)$ has extrema
- Minima of $V(x)$ are centres and maxima are saddle points, as can be seen from linear approximation

Example 2: Qualitatively sketch the phase paths of the conservative system with potential energy illustrated below.



Damping & nonlinear stiffness:

In a conservative system, changing the stiffness manifests in $V(x)$, addition of (positive) damping has the effect of stabilising.

Often this manifests in those critical points that were centres for the undamped system becoming asymptotically stable spirals for the damped system, (e.g. damped pendulum in last lecture).

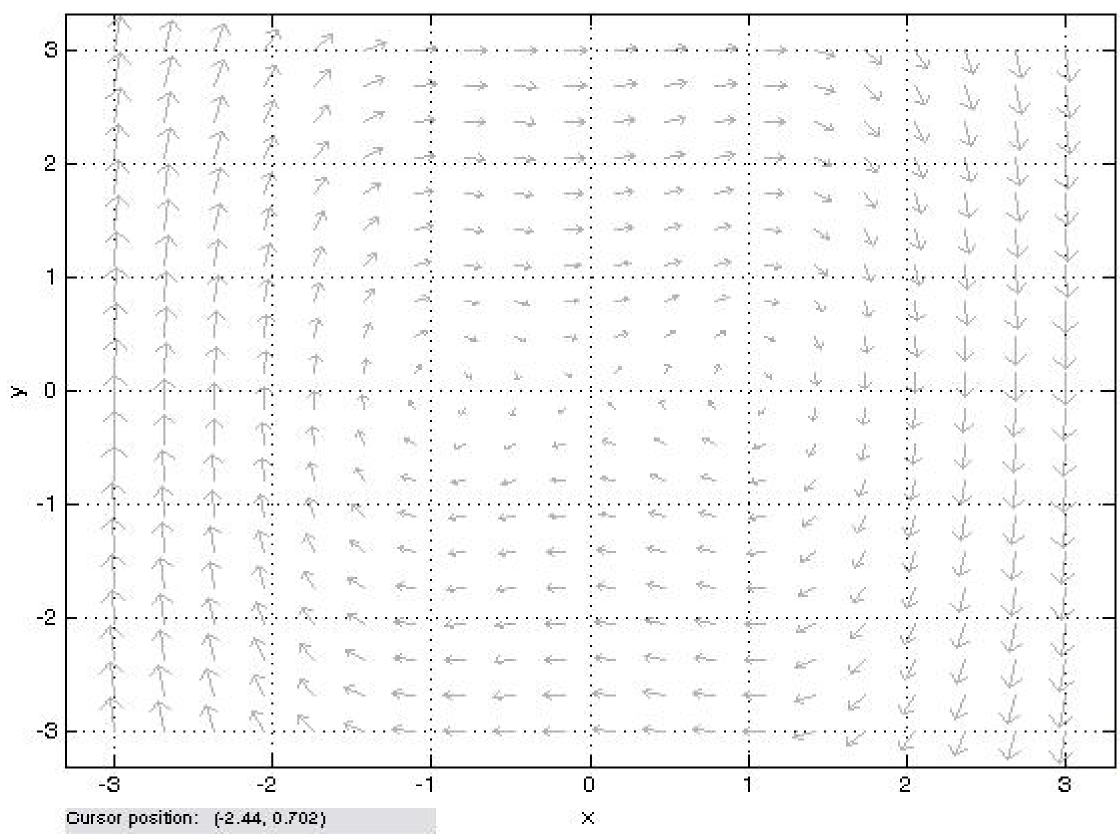
Example 3: Duffing's oscillator with negative linear stiffness

$$x' = y$$

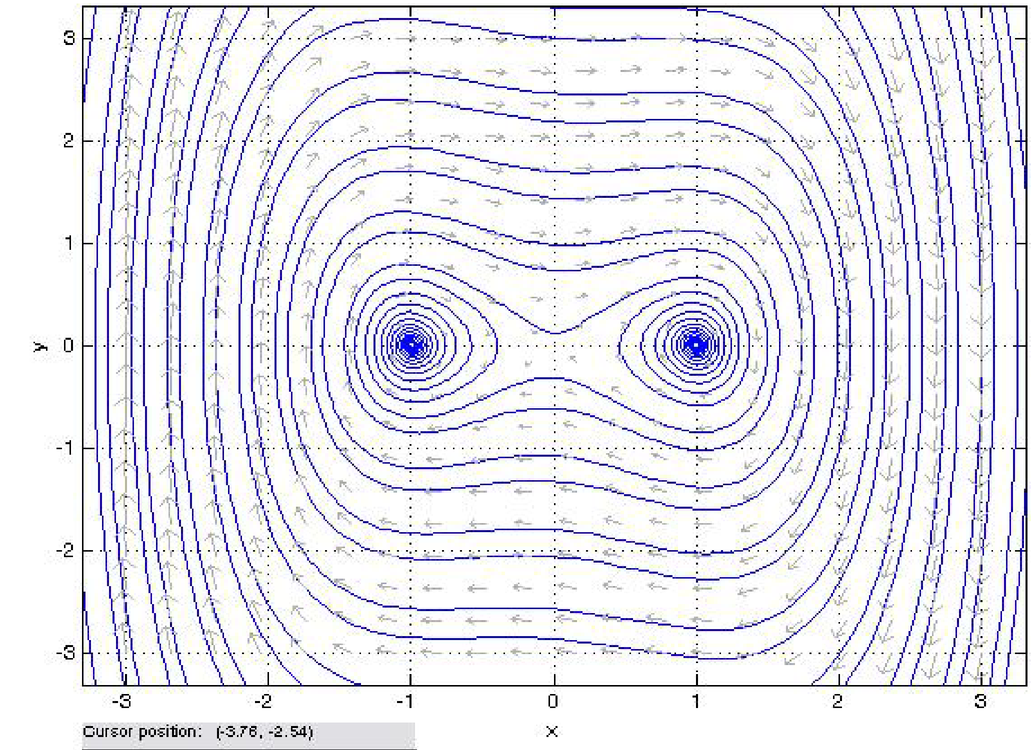
$$y' = -\frac{kx + ay + lx^3}{m}$$

(If $k > 0$ and $l = 0$, this is a linear oscillator & damped for $a > 0$)

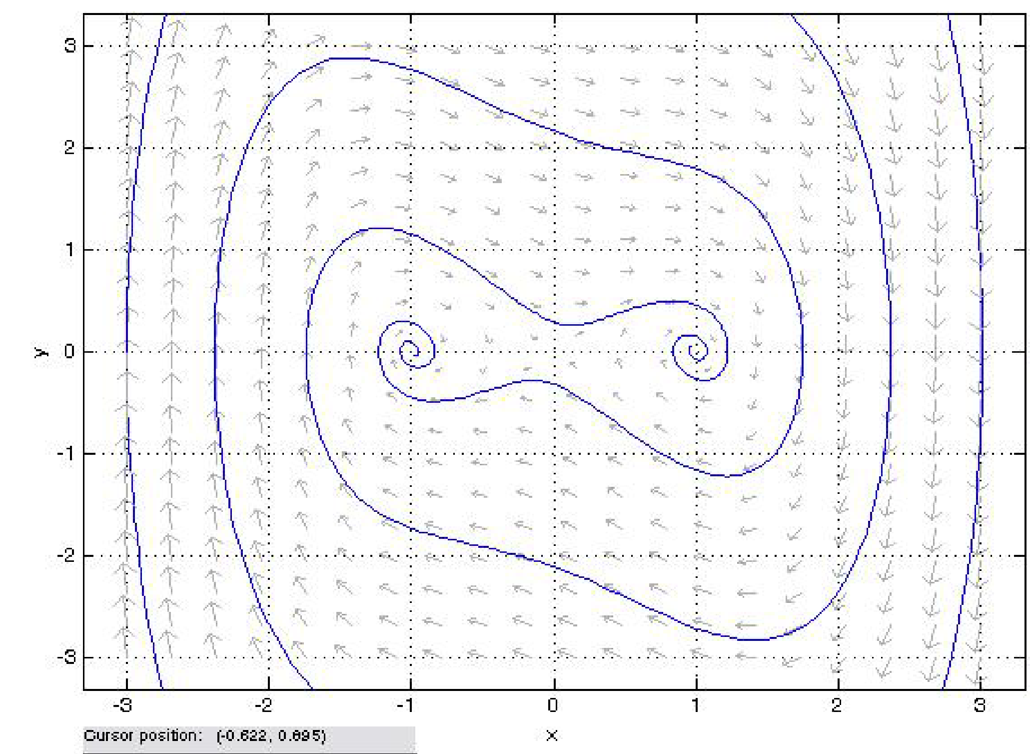
$k = -1, a = 0, l = 1, m = 1$: undamped oscillator



$k = -1, a = 0.1, l = 1, m = 1$: two phase paths plotted



$k = -1, a = 0.5, l = 1, m = 1$: two phase paths plotted

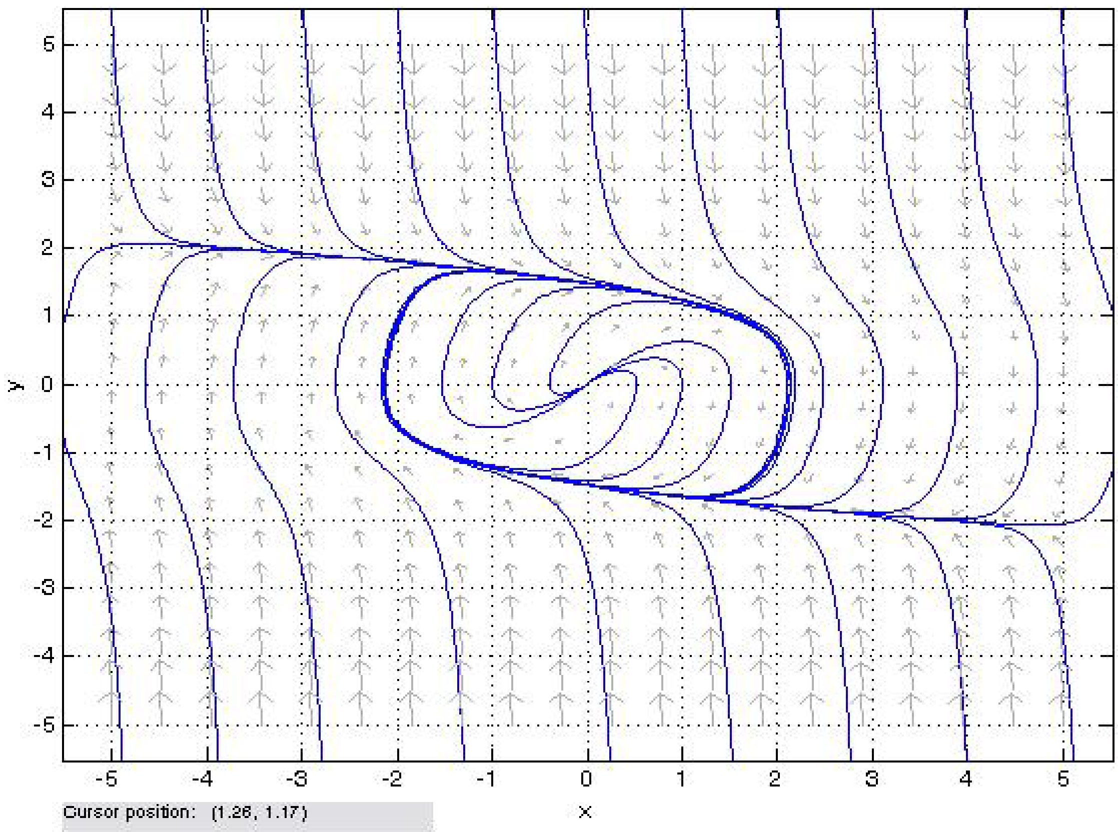


Example 4: Van der Pol's oscillator

$$x' = y$$
$$y' = -x + My - y^3$$

This is a linear oscillator with negative linear damping, but positive nonlinear damping.

For large enough velocities, we expect decay of the energy, but not for small velocities – what happens?



This stable structure is called a **limit cycle** and is just a stable periodic solution, (we can also have unstable limit cycles)

Energy interpretation of damped conservative systems:

The previous two oscillators were of general form:

$$\begin{aligned} x' &= y \\ y' &= F(x) - c(y) \end{aligned} \qquad (4)$$

where $c(y)$ denotes the damping function.

If we multiply the second equation in (4) by y and use the first equation, we have:

$$y'y = F(x)x' - c(y)y \qquad (5)$$

We integrate (5) with respect to t :

$$\frac{y^2}{2} + V(x) + \int^t c(y(s))y(s)ds = E + \int^t c(y(s))y(s)ds = \text{constant}$$

Thus, our energy conservation is modified by the damping term.

Now if we differentiate with respect to t , we see that (5) in fact means that

$$\frac{dE}{dt} = -c(y)y$$

i.e. the total energy along a stream path is reduced, by the damping, at rate $-c(y)y$.

- Duffings equation: $c(y)y = ay^2 > 0$, and we saw that the solutions all eventually spiralled into one of two asymptotically stable critical points. As we increased a the decay rate of the energy grew & we spiral inwards faster.
- Van der Pol's oscillator: $c(y)y = -My^2 + y^4$. For large y we lose energy, but for small y we gain energy! Thus, all large amplitudes decay, but not to zero. The limit cycle represents the path where the net increase in energy balances the decrease, over one cycle.